1. Let \( f : S_n \to G \) be any homomorphism (to some group \( G \)) such that \( f(1 2) = e \). Prove that \( f(x) = e \) for all \( x \).

**Solution:** The kernel of \( f \) is a normal subgroup in \( S_n \) containing \((1 2)\). Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole \( S_n \), so \( f(x) = e \) for all \( x \).

2. Are the following subsets of \( D_n \) subgroups? Normal subgroups?
   a) All reflections in \( D_n \)
   b) All rotations in \( D_n \)
   c) \( \{1, s\} \) where \( s \) is some reflection

**Solution:** (a) No: it does not contain the identity! (b) Yes, it is a normal subgroup. It contains the identity, the product of two rotations is a rotation and the inverse of a rotation is a rotation, so it is a subgroup. Also, \( \det(g^{-1}xg) = \det(x) \), so any matrix conjugate to a rotation must have determinant 1 and hence is a rotation, so it is normal. (c) It is clearly a cyclic subgroup generated by \( s \), but it is not normal: if \( s_1 \) is some other reflection then one can check (see also problem 6) that \( (s_1)^{-1}ss_1 \neq s \).

3. Consider the set
   \[ G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\} \]
   and a function \( f : G \to \mathbb{R}^* \),
   \[ f \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = a. \]

   a) Prove that \( G \) is a subgroup of \( GL_2 \)
   b) Prove that \( f \) is a homomorphism.
   c) Find the kernel and image of \( f \).

**Solution:** a) We have to check 3 defining properties a subgroup:

- Identity is in \( G \): take \( a = 1, b = 0 \)
• Closed under multiplication:
\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}
\]

• Closed under taking inverses: we need \( aa_1 = 1, ab_1 + b = 0 \), so
\[
a_1 = 1/a, \ b_1 = -b/a.
\]

b) From (a) we see that \( f(AB) = aa_1 = f(A)f(B) \).
c) Since \( a \) can be arbitrary, \( Im(f) = \mathbb{R}^* \). Now
\[
Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}.
\]

4. Consider the permutation
\[
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}
\]
a) Decompose \( f \) into non-intersecting cycles
b) Find the order of \( f \)
c) Find the sign of \( f \)
d) Compute \( f^{-1} \)

**Solution:** \( f = (1 \ 5 \ 3)(2 \ 6)(4 \ 7) \), it has order \( lcm(3,2,2) = 6 \) and sign \((-1)^{3-1}(-1)(-1) = 1\), \( f^{-1} = (1 \ 3 \ 5)(2 \ 6)(4 \ 7) \).

5. Find all possible orders of elements in \( D_6 \).

**Solution:** The group contains reflections and rotations by multiples of \( 360^\circ/6 = 60^\circ \). Every reflection has order 2, identity has order 1, rotations by \( 60^\circ \) and \( 300^\circ \) have order 6, rotations by \( 120^\circ \) and \( 240^\circ \) have order 3 and rotation by \( 180^\circ \) has order 2.

6. A soccer ball has 32 faces: 12 are regular pentagons and 20 are regular hexagons. Every pentagon is surrounded by 5 hexagons, while every hexagon neighbors 3 pentagons and 3 hexagons. Consider the action of isometry group of this ball on faces:
(a) Find the orbit and stabilizer of a pentagonal face
(b) Compute the size of the isometry group
(c)* Find the orbit and stabilizer of a hexagonal face
Solution: (a) The orbit of a pentagon consists of all pentagons and has 12 elements. The stabilizer of a pentagon is the dihedral group $D_5$ and has 10 elements.

(b) By counting formula, the isometry group of the ball has $12 \times 10 = 120$ elements.

(c) The orbit of a hexagon consists of all hexagons and has 20 elements. The stabilizer of a hexagon consists of all elements in $D_6$ which send neighboring hexagons to hexagons and pentagons to pentagons. It contains rotations by $0, 2\pi/3$ and $4\pi/3$, as well as reflections in three planes through midpoints of opposite edges. Therefore the stabilizer has 6 elements, and we can check by counting formula that $20 \times 6 = 120$.

7. The trace of a $2 \times 2$ matrix is defined as

$$\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$ 

a) Prove that $\text{tr}(AB) = \text{tr}(BA)$ for all $A$ and $B$

b) Prove that $\text{tr}(A^{-1}XA) = \text{tr}(X)$, so the conjugate matrices have the same trace.

Solution: a) We have

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{pmatrix}.$$
so \( \text{tr}(AB) = aa_1 + bc_1 + cb_1 + dd_1 \). Clearly, this expression will not change if we swap \( A \) and \( B \).

b) By part (a), \( \text{tr}(A^{-1} \cdot XA) = \text{tr}(XA \cdot A^{-1}) = \text{tr}(X) \).

8. Prove that the equation \( x^2 + 1 = 4y \) has no integer solutions.

**Solution:** Let us consider all possible remainders of \( x \) modulo 4. If \( x \equiv 0 \pmod{4} \) then \( x^2 + 1 \equiv 1 \pmod{4} \); if \( x \equiv 1 \pmod{4} \) then \( x^2 + 1 \equiv 2 \pmod{4} \); if \( x \equiv 2 \pmod{4} \) then \( x^2 + 1 \equiv 1 \pmod{4} \); if \( x \equiv 3 \pmod{4} \) then \( x^2 + 1 \equiv 2 \pmod{4} \). Therefore \( x^2 + 1 \) is never divisible by 4.

9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

**Solution:** (a) Yes, for example \( D_3 \) and \( \mathbb{Z}_6 \). The orders of elements in \( D_3 \) are 1,2,3, while \( \mathbb{Z}_6 \) has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element \( x \) divides 7, so it should be 1 (and \( x = e \)) or 7. If the order of \( x \) equals 7, then this is just the cyclic group generated by \( x \). So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example \( D_4 \) and \( \mathbb{Z}_8 \). The orders of elements in \( D_4 \) are 1,2,4, while \( \mathbb{Z}_8 \) has an element of order 8, so they are not isomorphic.

10. (a) Prove that any homomorphism from \( \mathbb{Z}_{11} \) to \( S_{10} \) is trivial.

(b) Find a nontrivial homomorphism from \( \mathbb{Z}_{11} \) to \( S_{11} \).

**Solution:** (a) Suppose that \( f : \mathbb{Z}_{11} \rightarrow S_{10} \) is a homomorphism. Then by Counting Formula \( |\text{Im}(f)| \) divides \( |\mathbb{Z}_{11}| = 11 \). Since 11 is prime, the image of \( f \) has either 1 or 11 elements.

On the other hand, by Lagrange Theorem \( |\text{Im}(f)| \) divides \( |S_{10}| = 10! \). Since 11 does not divide 10!, the image must have 1 element, so \( f \) is trivial.

(b) We can define \( f(k) = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^k \) for all integer \( k \). Then \( f(k + l) = f(k)f(l) \) and \( f(11) = e \), so \( f \) defines a homomorphism from \( \mathbb{Z}_{11} \) to \( S_{11} \).

11. How many conjugacy classes are there in \( S_5 \)?

**Solution:** The conjugacy classes in \( S_n \) correspond to cycle types. There are 7 possible cycle types (listed by length of their cycles): \( e, 2, 3, 4, 5, 2 + 2, 2 + 2 + 3 \).

12. Are the following matrices orthogonal? Do they preserve orientation?
\[
\begin{pmatrix}
1 & -1 \\
0 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\
\end{pmatrix}.
\]

**Solution:** A matrix is orthogonal if \(A^tA = I\), and preserves orientation if \(\det(A) > 0\), so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every \(n\) there is a group with \(n\) elements.

**Solution:** Indeed, consider cyclic group \(\mathbb{Z}_n\).

14. Solve the system of equations
\[
\begin{align*}
x &= 1 \mod 8 \\
x &= 3 \mod 6.
\end{align*}
\]

**Solution:** We cannot apply Chinese Remainder Theorem since 6 and 8 are not coprime. Nevertheless, the remainders of \(x \mod 6\) and \(x \mod 8\) do not change if we add \(\text{LCM}(6, 8) = 24\) to \(x\), so we can consider \(x \mod 24\). Since \(x = 1 \mod 8\), we get
\[
x = 1, 9, 17 \mod 24
\]
Among these values, only \(x = 9\) is equal to 3 modulo 6.

**Answer:** \(x = 9 \mod 24\).

15. Compute \(3^{100} \mod 7\).

**Solution:** We have
\[
3^1 = 3, \quad 3^2 = 2, \quad 3^3 = 2 \cdot 3 = 6, \quad 3^4 = 6 \cdot 3 = 4, \quad 3^5 = 4 \cdot 3 = 5, \quad 3^6 = 5 \cdot 3 = 1 \mod 7
\]
Therefore
\[
3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7.
\]

16. Find the conjugacy classes and centralizers of all elements in the dihedral group (a) \(D_5\) (b) \(D_6\).

**Solution:** (a) We proved in class that the conjugacy classes in \(D_5\) have the following form:
\[
\{e\}, \{\text{rotations by } \pm \frac{2\pi}{5}\}, \{\text{rotations by } \pm \frac{4\pi}{5}\}, \{\text{all reflections}\}.
\]

5
The centralizer of $e$ is the whole group $D_5$. The centralizer of any nontrivial rotation consists of all rotations and has size 5. The centralizer of any reflection has size 2 and consists of this reflection and identity. We can check the counting formula

$$(\text{size of conjugacy class}) \times (\text{size of centralizer}) = (\text{size of } D_5) = 10 :$$

$$1 \cdot 10 = 2 \cdot 5 = 5 \cdot 2 = 10.$$

(b) We proved in class that the conjugacy classes in $D_6$ have the following form:

$\{e\}, \{\text{rotations by } \pm \frac{2\pi}{6}\}, \{\text{rotations by } \pm \frac{4\pi}{6}\}, \{\text{rotation by } \pi\},$

$\{\text{reflections in lines through opposite vertices}\},$

$\{\text{reflections in lines through idpoints of opposite sides}\}.$

The centralizer of $e$ and of rotation by $\pi$ is the whole group $D_6$. The centralizer of any other rotation consists of all rotations and has size 6. The centralizer of any reflection has size 4 and consists of this reflection, reflection in perpendicular line, rotation by $\pi$ and identity. We can check the counting formula

$$1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 12.$$