MAT 150A, Fall 2018
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.

Solution: The kernel of f is a normal subgroup in S_n containing $(1\ 2)$. Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole S_n, so $f(x) = e$ for all x.

2. Are the following subsets of D_n subgroups? Normal subgroups?
a) All reflections in D_n
b) All rotations in D_n
c) $\{1, s\}$ where s is some reflection

Solution: (a) No: it does not contain the identity! (b) Yes, it is a normal subgroup. It contains the identity, the product of two rotations is a rotation and the inverse of a rotation is a rotation, so it is a subgroup. Also, $\det(g^{-1}xg) = \det(x)$, so any matrix conjugate to a rotation must have determinant 1 and hence is a rotation, so it is normal. (c) It is clearly a cyclic subgroup generated by s, but it is not normal: if s_1 is some other reflection then one can check (see also problem 6) that $(s_1)^{-1}ss_1 \neq s$.

3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.

Solution: a) We have to check 3 defining properties a subgroup:

- Identity is in G: take $a = 1, b = 0$

- Closed under multiplication:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}$$

- Closed under taking inverses: we need $aa_1 = 1, ab_1 + b = 0$, so

$$a_1 = 1/a, \ b_1 = -b/a.$$

b) From (a) we see that $f(AB) = aa_1 = f(A)f(B)$.
c) Since a can be arbitrary, $Im(f) = \mathbb{R}^*$. Now

$$Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}.$$

4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

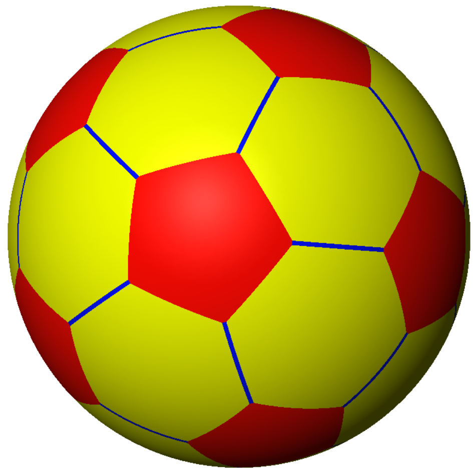
a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}

Solution: $f = (1\ 5\ 3)(2\ 6)(4\ 7)$, it has order $lcm(3, 2, 2) = 6$ and sign $(-1)^{3-1}(-1)(-1) = 1$, $f^{-1} = (1\ 3\ 5)(2\ 6)(4\ 7)$.

5. Find all possible orders of elements in D_6.

Solution: The group contains reflections and rotations by multiples of $360°/6 = 60°$. Every reflection has order 2, identity has order 1, rotations by $60°$ and $300°$ have order 6, rotations by $120°$ and $240°$ have order 3 and rotation by $180°$ has order 2.

6. A soccer ball has 32 faces: 12 are regular pentagons and 20 are regular hexagons. Every pentagon is surrounded by 5 hexagons, while every hexagon neighbors 3 pentagons and 3 hexagons. Consider the action of isometry group of this ball on faces:
(a) Find the orbit and stabilizer of a pentagonal face
(b) Compute the size of the isometry group
(c)* Find the orbit and stabilizer of a hexagonal face



Solution: (a) The orbit of a pentagon consists of all pentagons and has 12 elements. The stabilizer of a pentagon is the dihedral group D_5 and has 10 elements.

(b) By counting formula, the isometry group of the ball has $12 \times 10 = 120$ elements.

(c) The orbit of a hexagon consists of all hexagons and has 20 elements. The stabilizer of a hexagon consists of all elements in D_6 which send neighboring hexagons to hexagons and pentagons to pentagons. It contains rotations by $0, 2\pi/3$ and $4\pi/3$, as well as reflections in three planes through midpoints of opposite edges. Therefore the stabilizer has 6 elements, and we can check by counting formula that $20 \times 6 = 120$.

7. The *trace* of a 2×2 matrix is defined as

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all A and B

b) Prove that $\mathrm{tr}(A^{-1}XA) = \mathrm{tr}(X)$, so the conjugate matrices have the same trace.

Solution: a) We have

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{pmatrix},$$

so $\mathrm{tr}(AB) = aa_1 + bc_1 + cb_1 + dd_1$. Clearly, this expression will not change if we swap A and B.

b) By part (a), $\mathrm{tr}(A^{-1} \cdot XA) = \mathrm{tr}(XA \cdot A^{-1}) = \mathrm{tr}(X)$.

8. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.

Solution: Let us consider all possible remainders of x modulo 4. If $x = 0 \mod 4$ then $x^2 + 1 = 1 \mod 4$; if $x = 1 \mod 4$ then $x^2 + 1 = 2 \mod 4$;if $x = 2 \mod 4$ then $x^2 + 1 = 1 \mod 4$;if $x = 3 \mod 4$ then $x^2 + 1 = 2 \mod 4$. Therefore $x^2 + 1$ is never divisible by 4.

9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

Solution: (a) Yes, for example D_3 and $\mathbb{Z}_6$. The orders of elements in D_3 are 1,2,3, while $\mathbb{Z}_6$ has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element x divides 7, so it should be 1 (and $x = e$) or 7. If the order of x equals 7, then this is just the cyclic group generated by x. So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example D_4 and $\mathbb{Z}_8$. The orders of elements in D_4 are 1,2,4, while $\mathbb{Z}_8$ has an element of order 8, so they are not isomorphic.

10. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to S_{10} is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

Solution: (a) Suppose that $f : \mathbb{Z}_{11} \to S_{10}$ is a homomorphism. Then by Counting Formula $|Im(f)|$ divides $|Z_{11}| = 11$. Since 11 is prime, the image of f has either 1 or 11 elements.

On the other hand, by Lagrange Theorem $|Im(f)|$ divides $|S_{10}| = 10!$. Since 11 does not divide 10!, the image must have 1 element, so f is trivial.

(b) We can define $f(k) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)^k$ for all integer k. Then $f(k + l) = f(k)f(l)$ and $f(11) = e$, so f defines a homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

11. How many conjugacy classes are there in S_5?

Solution: The conjugacy classes in S_n correspond to cycle types. There are 7 possible cycle types (listed by length of their cycles): $e, 2, 3, 4, 5, 2 + 2, 2 + 3$.

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{pmatrix}.$$

Solution: A matrix is orthogonal if $A^tA = I$, and preserves orientation if $\det(A) > 0$, so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every n there is a group with n elements.

Solution: Indeed, consider cyclic group $\mathbb{Z}_n$.

14. Solve the system of equations

$$\begin{cases} x = 1 \mod 8 \\ x = 3 \mod 6. \end{cases}$$

Solution: We cannot apply Chinese Remainder Theorem since 6 and 8 are not coprime. Nevertheless, the remainders of $x \mod 6$ and $\mod 8$ do not change if we add $LCM(6, 8) = 24$ to x, so we can consider $x \mod 24$. Since $x = 1 \mod 8$, we get

$$x = 1, 9, 17 \mod 24$$

Among these values, only $x = 9$ is equal to 3 modulo 6.

Answer: $x = 9 \mod 24$.

15. Compute $3^{100} \mod 7$.

Solution: We have

$$3^1 = 3,\ 3^2 = 2,\ 3^3 = 2{\cdot}3 = 6,\ 3^4 = 6{\cdot}3 = 4,\ 3^5 = 4{\cdot}3 = 5,\ 3^6 = 5{\cdot}3 = 1 \mod 7$$

Therefore

$$3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7.$$

16. Find the conjugacy classes and centralizers of all elements in the dihedral group (a) D_5 (b) D_6.

Solution: (a) We proved in class that the conjugacy classes in D_5 have the following form:

$$\{e\}, \{\text{rotations by } \pm\frac{2\pi}{5}\}, \{\text{rotations by } \pm\frac{4\pi}{5}\},\ \{\text{all reflections}\}.$$

The centralizer of e is the whole group D_5. The centralizer of any nontrivial rotation consists of all rotations and has size 5. The centralizer of any reflection has size 2 and consists of this reflection and identity. We can check the counting formula

$$\text{(size of conjugacy class)} \times \text{(size of centralizer)} = \text{(size of } D_5) = 10:$$

$$1 \cdot 10 = 2 \cdot 5 = 2 \cdot 5 = 5 \cdot 2 = 10.$$

(b) We proved in class that the conjugacy classes in D_6 have the following form:

$$\{e\}, \{\text{rotations by } \pm\frac{2\pi}{6}\}, \{\text{rotations by } \pm\frac{4\pi}{6}\}, \quad \{\text{rotation by } \pi\},$$

$$\{\text{reflections in lines through opposite vertices}\},$$

$$\{\text{reflections in lines through idpoints of opposite sides}\}.$$

The centralizer of e and of rotation by π is the whole group D_6. The centralizer of any other rotation consists of all rotations and has size 6. The centralizer of any reflection has size 4 and consists of this reflection, reflection in perpendicular line, rotation by π and identity. We can check the counting formula

$$1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 12.$$