5.1 (25 points) Let \( \varphi : G \to G' \) be a surjective homomorphism. Prove that if \( G \) is cyclic then \( G' \) is cyclic, and if \( G \) is abelian, then \( G' \) is abelian.

**Proof:** Suppose that \( G \) is cyclic with generator \( g \), so every element of \( G \) has the form \( g^k \) for some \( k \). Since \( \varphi \) is surjective, every element of \( G' \) has the form \( \varphi(z) \) for some \( z \). Then \( z = g^k \) and

\[
\varphi(z) = \varphi(g^k) = \varphi(g \cdots g) = \varphi(g) \cdots \varphi(g) = \varphi(g)^k.
\]

Therefore \( G' \) is a cyclic group generated by \( \varphi(g) \).

Suppose that \( G \) is abelian, and let \( x, y \in G' \). Since \( \varphi \) is surjective, we have \( x = \varphi(z) \) and \( y = \varphi(w) \) for some \( z, w \in G \). Now

\[
x \cdot y = f(z) \cdot f(w) = f(zw), \quad y \cdot x = f(w) \cdot f(z) = f(wz).
\]

Since \( G \) is abelian, we have \( zw = wz \), so \( f(zw) = f(wz) \) and \( xy = yx \). Therefore \( G' \) is abelian.

6.2. (25 points) Describe all homomorphisms \( \varphi : \mathbb{Z} \to \mathbb{Z} \). Determine which are injective, which are surjective and which are isomorphisms.

**Solution:** If \( \phi \) is an homomorphism then \( \varphi(0) = 0 \), and for all \( x, y \) \( \varphi(x + y) = \varphi(x) + \varphi(y) \). Suppose that \( \varphi(1) = n \), then \( \varphi(2) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = 2n \). Similarly (one can prove this by induction), for all \( k > 0 \) \( \varphi(k) = kn \). Now \( \varphi(-k) + \varphi(k) = \varphi(0) = 0 \), so \( \varphi(-k) = -\varphi(k) = -kn \). Therefore for all \( x \) we have \( \phi(x) = nx \). Indeed, such function is a homomorphism.

It is injective if and only if \( \ker \varphi = \{0\} \), that is, if and only if \( n \neq 0 \). Since all elements in the image of \( \varphi \) are divisible by \( n \), it is surjective for \( n = \pm 1 \). As a result, \( \varphi \) is an isomorphism if and only if \( n = \pm 1 \), so \( \varphi(x) = x \) or \( \varphi(x) = -x \).

9.3. (25 points) Prove that every integer is congruent to the sum of its decimal digits modulo 9.

**Solution:** Consider a number \( a \) with digits \( a_1, \ldots, a_n \). We have

\[
a = a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \ldots + a_n = a_1 + \ldots + a_n \mod 9,
\]

since \( 10^k = 1 \mod 9 \) for all \( k \). Indeed, \( 10 = 1 \mod 9 \), so \( 10^k = 1^k = 1 \mod 9 \).

9.4. (25 points) Solve the congruence \( 2x = 5 \) modulo 9 and modulo 6.

**Solution:** If \( 2x = 5 \mod 6 \) then \( 2x = 5 + 6k \), but \( 2x \) and \( 6k \) are even while \( 5 \) is an odd number. Therefore the equation \( 2x = 5 \mod 6 \) has no solution.

To solve the equation \( 2x = 5 \mod 9 \), consider the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x \mod 9 )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that the only solution is \( x = 7 \mod 9 \).