8.5. (25 points) A finite group $G$ contains an element $x$ of order 10 and also an element $y$ of order 6. What can be said about the order of $G$?

Solution: By Lagrange Theorem, the order of $G$ is divisible by 10 and by 6, so it is divisible by $LCM(10, 6) = 30$.

8.6. (25 points) Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18, |G'| = 15$ and that $\varphi$ is not the trivial homomorphism. What is the order of the kernel?

Solution: By Counting Formula, $|\ker(\varphi)| \cdot |\text{Im}(\varphi)| = 18$, so $|\text{Im}(\varphi)|$ divides 18. Since $\text{Im}(\varphi)$ is a subgroup of $G'$, by Lagrange Theorem $|\text{Im}(\varphi)|$ also divides 15. Therefore $|\text{Im}(\varphi)|$ can be equal to 1 or 3. If $|\text{Im}(\varphi)| = 1$, then $\varphi$ is trivial. We conclude that $|\text{Im}(\varphi)| = 3$ and $|\ker(\varphi)| = 6$.

8.8. (25 points) Let $G$ be a group of order 25. Prove that $G$ has at least one subgroup of order 5, and that if it contains only one subgroup of order 5 then it is a cyclic group.

Solution: Let $g$ be an element in $G$, suppose that $g \neq e$. By Lagrange Theorem, the order of $g$ divides 25, so it could be equal to 1, 5 or 25. Since we assume that $g \neq e$, the order of $g$ is either 5 or 25. We have 2 cases:

(a) There is an element $g$ of order 25. Then the cyclic group $\langle g \rangle$ generated by $g$ has 25 elements and must coincide with $G$, so $G = \langle g \rangle$ is cyclic. It also contains a subgroup $\langle g^5 \rangle = \{e, g^5, g^{10}, g^{15}, g^{20}\}$ of order 5.

(b) All non-identity elements have order 5. Then $\langle g \rangle$ is a subgroup of order 5. Also, we can pick some element $y$ not in this subgroup, and then $\langle y \rangle$ is a different subgroup of order 5. So $G$ has more than one subgroup with 5 elements.

We conclude that there is always at least one subgroup with 5 elements, and if it is unique, we are in case (a) and $G$ is cyclic.

9.5. (25 points) Determine the integers $n$ such that the pair of congruences

\[ 2x - y = 1 \pmod n, \quad 4x + 3y = 2 \pmod n \]

has a solution.

Solution: From the first equation we can write $y = 2x - 1 \pmod n$ and plug into the second equation:

\[ 4x + 3(2x - 1) = 2, \quad 4x + 6x - 3 = 2, \quad 10x = 5 \pmod n. \]

Therefore for some $q$ we have

\[ 10x = qn + 5. \]

If $n$ is even, this equation has no solutions since $10x$ is even and $qn + 5$ is odd. If $n$ is odd, then we can write $n = 2k + 1$, and pick $x = k + 1$. Then

\[ 10x = 10k + 10 = 5n + 5 = 5 \pmod n, \]

and the equation is satisfied. Note that in this case $y = 2x - 1 = 2k + 1 = 0 \pmod n$. Therefore the system of equations has a solutions if and only if $n$ is odd.

Remark: One can also argue that for odd $n$ we have $GCD(2, n) = 1$, so the equation $2x = 1 \pmod n$ has a solution. By multiplying it by 5, we get $10x = 5$. 