Page 188: 1.1 (25 points) Determine all symmetries of figures 6.1.4, 6.1.6, and 6.1.7.

Solution: In 6.1.4 the group of isometries is generated by a glide symmetry $s$ (reflect in the horizontal line and translate by 1 unit). If we apply the glide symmetry twice, the reflections cancel, so $s^2$ is the translation by 2 units. Similarly, $s^3$ is a reflection with the translation by 3 units, and in general $s^{2k}$ is a translation by $2k$ units while $s^{2k+1}$ is a reflection composed with the translation by $(2k + 1)$ units.

In 6.1.6 there is a translational symmetry, and central symmetry (rotation by $\pi$) centered at the center of "n" or at the middle of the line connecting "yh".

In 6.1.7. the pattern is preserved by translations by an even number of units, reflection in the horizontal line composed with the translation by an odd number of units, as in 6.1.4. In addition, we can use reflection in a vertical line through the ”body” of one of the figures, and a central symmetry (rotation by $\pi$) at “*”.

3.2. (25 points) Let $m$ be an orientation-reversing isometry. Prove algebraically that $m^2$ is a translation.

Solution 1: By the classification theorem, $m$ is either a reflection or a glide reflection, that is, a reflection in a line $l$ followed by a translation by vector $v$ parallel to $l$. In the first case $m^2$ is identity, and in the second it is a translation by $2v$.

Solution 2: By definition, $m(x) = Ax + b$ where $A$ is an orthogonal $2 \times 2$ matrix. Since $m$ is orientation-reversing, det $A = -1$ and $A$ is a reflection, so $A^2 = I$. Then

$$m^2(x) = m(m(x)) = A(Ax + b) + b = A^2x + Ab + b = x + Ab + b,$$

so $m^2$ is a translation by vector $Ab + b$.

5.5. (25 points) Prove that the group of symmetries of the frieze pattern $\triangleleft \triangleleft \triangleleft \triangleleft \triangleleft \triangleleft$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$. 
Solution: Let $s$ be a reflection in the $x$-axis, and let $t$ be a translation to the right by 1 unit. Then $st = ts$ is the glide reflection, $t^2$ is a translation by 2 units etc. Every isometry has the form $s^at^b$ (since $st = ts$), and since $s^2 = e$ we get $a = 0$ or $a = 1$. We can identify this with $\mathbb{Z}_2 \times \mathbb{Z}$ by writing $(0,k) = t^k$ and $(1,k) = st^k$.

Problem A: (25 points) Let $x$ and $y$ be the reflections in two lines with angle $\pi/n$ between them.

(a) Prove that $x^2 = y^2 = (xy)^n = I$.

(b) Prove that $x$ and $y$ generate the dihedral group $D_n$, that is, every element of $D_n$ can be presented as a product of $x$ and $y$ in some order.

Solution: a) Since $x$ and $y$ are reflections, we have $x^2 = y^2 = I$. Also, their composition $xy$ is a rotation by angle $2\pi/n$, so $(xy)^n$ is a rotation by $2\pi$ which is equivalent to identity. So $(xy)^n = I$.

b) Since $xy$ is a rotation by angle $2\pi/n$, $(xy)^k$ is the rotation by $k \cdot 2\pi/n$, so all rotations in $D_n$ can be written as $(xy)^k$. Let us prove that all reflections in $D_n$ can be written as $y(xy)^k$.

Indeed, let us label the reflections in $D_n$ by $z_1, \ldots, z_n$ such that $z_1 = y$ and the angle between $z_i$ and $z_{i+1}$ equals $\pi/n$. Then for all $i$ the composition $z_iz_{i+1}$ is the rotation by $2\pi/n$, and

$$xy = z_1z_2 = z_2z_3 = \ldots = z_{i}z_{i+1} = \ldots$$

Now $z_1 = y$, so $z_2 = z_1(xy) = y(xy), z_3 = z_2(xy) = y(xy)^2$ and so on. Therefore all $z_k$ can be written as products of $x$ and $y$, so $x$ and $y$ generate $D_n$.

Remark: One can also argue that $y(xy)^k$ are all distinct for $0 \leq k \leq n-1$ and all are contained in $D_n$, so they exhaust all reflections in $D_n$. 

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