Page 223: 5.2. (25 points) What is the centralizer of the element $(1\ 2)$ in $S_5$?

Solution: A permutation $f$ commutes with $(1\ 2)$ if all cycles in $f$ either coincide with $(1\ 2)$ or do not intersect $(1\ 2)$. Therefore the centralizer of $(1\ 2)$ has 12 elements:

\[
\{e, (3\ 4), (3\ 5), (4\ 5), (3\ 4\ 5), (3\ 5\ 4), \\
(1\ 2), (1\ 2)(3\ 4), (1\ 2)(3\ 5), (1\ 2)(4\ 5), (1\ 2)(3\ 4\ 5), (1\ 2)(3\ 5\ 4)\}.
\]

It is isomorphic to $S_3 \times \mathbb{Z}_2$.

5.5. (25 points) Let $p$ and $q$ be permutations. Prove that the products $pq$ and $qp$ have cycles of equal sizes.

Solution: Observe that $qp = q(pq)q^{-1}$, so $pq$ and $qp$ are conjugate in $S_n$, therefore they have cycles of equal sizes.

Problem A: (25 points) a) A brick has height 1, width 2 and length 3. How many isometries does it have? Present each isometry by an orthogonal $3 \times 3$ matrix, assuming that the center of the brick has coordinates $(0, 0, 0)$ and the sides are parallel to the coordinate axis.

b) Find the orbit and stabilizer of this action for each face of the brick.

c) Find the orbit and stabilizer of this action for each vertex of the brick.

d) Find the orbit and stabilizer of this action for each edge of the brick.

Solution: a) The isometry group has 8 elements: identity

\[
I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

three reflections in coordinate planes:

\[
\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
three rotations by $\pi$ around coordinate axis:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

and the central symmetry

$$
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

b) The orbit of a face has size 2. It contains this face and the opposite face, since all other faces have different area. The stabilizer of a face contains two reflections, identity, and rotation by $\pi$. For example, the stabilizer of the top face contains reflections in $xz$ and $yz$ planes, identity, and rotation by $\pi$ around $z$-axis. We have $2 \times 4 = 8$.

c) All 8 vertices are in the same orbit, and the stabilizer of a vertex is trivial. We have $1 \times 8 = 8$.

d) The orbit of an edge contains 4 edges of the same length. The stabilizer of an edge contains identity and one reflection. For example, the stabilizer of a vertical edge contains identity and reflection in $xy$ plane. We have $4 \times 2 = 8$.

**Problem B:** (25 points) Consider the action of the group $S_7$ on the set of 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$.

a) Find the stabilizer of the subset $\{1, 2, 3\}$.

b) Describe the orbit of $\{1, 2, 3\}$ and use Counting Formula to compute the size of this orbit.

**Solution:** a) A permutation $f$ sends $\{1, 2, 3\}$ to $\{f(1), f(2), f(3)\}$. Therefore the stabilizer of $\{1, 2, 3\}$ contains all permutations in $S_7$ which permute 1,2,3 and 4,5,6,7 separately. It is isomorphic to $S_3 \times S_4$ and has $3!4!$ elements.

b) The orbit of $\{1, 2, 3\}$ consists of all 3-element subsets. By Counting Formula, the size of this orbit (=the number of 3-element subsets) equals

$$
\binom{7}{3} = \frac{7!}{3!4!}.
$$