MAT 150A, Fall 2021
Practice problems for Midterm 2

1. Prove that every group with 4 elements has an element of order 2.
2. Are the following matrices orthogonal? Do they preserve orientation? Describe the corresponding transformations geometrically.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. Let A be the counterclockwise rotation of the plane by $90°$, let B be the reflection in the line $\{x = y\}$. Present the transformation A, B, AB, BA by matrices, describe AB and BA geometrically.
4. Are the following functions homomorphisms?
(a) $f : \mathbb{R}^* \to \mathbb{R}^*, f(x) = x + 1$.
(b) $f : \mathbb{R}^* \to \mathbb{R}^*, f(x) = 1/x$
5. Prove that the groups $\mathbb{Z}_6$ and S_3 are not isomorphic.
6. Is it possible to construct a surjective homomorphism from a group with 6 elements to a group with (a) 7 elements (b) 5 elements (c) 3 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.
7. Is it possible to construct an injective homomorphism from a group with 6 elements to a group with (a) 3 elements (b) 9 elements (c) 12 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.
8. Solve the system of equations

$$\begin{cases} x = 3 \mod 5 \\ x = 4 \mod 6. \end{cases}$$

9. Is there an element of order 2 in (a) $(\mathbb{Z}_9, +)$? (b) $(\mathbb{Z}_9^*, \times)$ (c) $(\mathbb{Z}_{99}, +)$? (d) $(\mathbb{Z}_{99}^*, \times)$?
10. If we label the diagonals in the square by 1 and 2, every isometry of the square would permute them. This gives a homomorphism from D_4 to S_2. Describe its kernel and image.
11*. Color alternate vertices of the regular 6-gon in black and white. Every element of D_6 either preserves all colors or changes all of them, this defines a homomorphism from D_6 to S_2. Describe its kernel and image.