MAT 150A, Fall 2021
Solutions to homework 3

1. Let x and y be elements of a group G. Assume that each of the elements x, y and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G.

Solution: Let us first prove that $xy = yx$. Since x, y and xy have order 2, we have

$$xx = 1, yy = 1, xyxy = 1.$$

If we multiply the last equation by x on the left and by y on the right, we get

$$xy = x(xyxy)y = (xx)yx(yy) = yx.$$

Now we can check that H is a subgroup: it contains 1 and each element is its own inverse, so we just need to check that it is closed under multiplication. Since $xy = yx$, any two elements of H commute and we can write

$$yx = xy,\ x(xy) = y = (xy)x,\ y(xy) = x = (xy)y.$$

2. Prove that every integer is congruent to the sum of its decimal digits modulo 9.

Solution: Suppose that a number N has digits $a_1, \ldots, a_k$, read from left to right. Then

$$N = a_1 \cdot 10^{k-1} + \ldots + a_{k-1} \cdot 10 + a_k.$$

Now $10 = 1 \mod 9$, so $10^i = 1^i = 1 \mod 9$ for all i, and

$$N = a_1 \cdot 1 + \ldots + a_{k-1} \cdot 1 + a_k = a_1 + \ldots + a_k \mod 9.$$

3. Solve the equation $2x = 5$ modulo 9 and modulo 6.

Solution: We can make the following tables

$x \mod 9$	0	1	2	3	4	5	6	7	8
$2x \mod 9$	0	2	4	6	8	1	3	5	7

Therefore the only solution for $2x = 5 \mod 9$ is $x = 7 \mod 9$.

Similarly, modulo 6 we get

$x \mod 9$	0	1	2	3	4	5
$2x \mod 9$	0	2	4	0	2	4

So there are no solutions to $2x = 5 \mod 6$.

4. Compute $2^{2021} \mod 7$.

Solution: We have $2^3 = 1 \mod 7$, so

$$2^k = \begin{cases} 2, & k = 3q + 1 \\ 4, & k = 3q + 2 \\ 1, & k = 3q. \end{cases}$$

Since $2021 = 3 \cdot 673 + 2$, we get

$$2^{2021} = (2^3)^{673} \cdot 2^2 = 1^{673} \cdot 4 = 4 \mod 7.$$