MAT 150A, Fall 2021
Solutions for homework 4

1. Solve the system of equations $x = 3 \mod 5, x = 10 \mod 11$.

Solution: Since $GCD(5, 11) = 1$, by Chinese Remainder Theoremthe solution is unique modulo 55. To find this solution, we write all possible elements satisfying the second equation $x = 10 \mod 11$:

$x \mod 55$	10	21	32	43	54
$x \mod 5$	0	1	2	3	4

So the solution is $x = 43 \mod 55$.

2. Is it possible to construct an injective homomorphism (a) from $\mathbb{Z}_3$ to $\mathbb{Z}_4$? (b) From S_3 to S_4?

Solution: (a) No. Suppose $\varphi : \mathbb{Z}_3 \to \mathbb{Z}_4$ is a homomorphism, and $\varphi(1) = a$. Then $\varphi(2) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = a + a = 2a$. Similarly $\varphi(3) = 3a$, but $3 = 0$ in $\mathbb{Z}_3$. So we get the equation $\varphi(0) = 0 = \varphi(3) = 3a$, so $3a = 0 \mod 4$. Therefore $a = 0 \mod 4$, so any homomorophism from $\mathbb{Z}_3$ to $\mathbb{Z}_4$ sends every element to 0 and is not injective.

(b) Yes, we can extend any permutation f of three elements to a permutation of 4 elements by $f(4) = 4$. Clearly, this is a homomorphism and it is injective.

3. A finite group G contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?

Solution: By Lagrange Theorem $|G|$ is divisible by 10 and by 6, so it is divisible by $LCM(10, 6) = 30$.

4. Let $\varphi : G_1 \to G_2$ be a group homomorphism. Suppose that $|G_1| = 18, |G_2| = 15$ and that φ is not the trivial homomorphism. What is the order of the kernel?

Solution: By Counting formula we have $|\operatorname{Ker}\varphi| \cdot |\operatorname{Im}\varphi| = |G_1| = 18$ and by Lagrange Theorem $|\operatorname{Im}\varphi|$ divides $|G_2| = 15$. Therefore $|\operatorname{Im}\varphi|$ divides both 15 and 18, so it is either equal to 3 or to 1. Since φ is not trivial, we get $|\operatorname{Im}\varphi| = 3$, and by Counting formula $|\operatorname{Ker}\varphi| = 6$.