MAT 150A, Fall 2021
Solutions to homework 5

1. Are the following matrices orthogonal?

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Solution: (a) We have

$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq I,$$

so A is not orthogonal.

(b) We have

$$A^T A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} `1 & 0 \\ 0 & 1 \end{pmatrix},$$

sp this matrix is orthogonal. In fact, this is the matrix of rotation by angle $\frac{\pi}{3}$:

$$\begin{pmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{pmatrix}$$

2. Find all diagonal 3×3 matrices

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

which are orthogonal.

Solution: We have

$$A^T A = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

so A is orthogonal if $a^2 = b^2 = c^2 = 1$, so $a = \pm 1, b = \pm 1, c = \pm 1$.

Recall that an orthogonal matrix A is called **orientation reversing** if $\det(A) = -1$ and **orientation preserving** if $\det(A) = 1$.

3. Find all 2×2 orientation reversing matrices of finite order.

Solution: By classification theorem, any 2×2 orthogonal orientation reversing matrix is a reflection, all reflections have order 2.

4. Find all 2×2 orientation preserving matrices of finite order.

Solution: By classification theorem, any 2×2 orthogonal orientationpreserving matrix A is a rotation by some angle φ. Its power A^n is a rotation by the angle $n\varphi$, so $A^n = I$ if and only if $n\varphi = 2\pi k$, or $\varphi = \frac{2\pi k}{n}$ for some integer k.