MAT 150A, Fall 2021
Solutions to homework 6

1. Let A and B be the reflections in two lines with angle $\frac{\pi}{n}$ between them.
(a) Prove that $A^2 = B^2 = (AB)^n = I$.
(b) Prove that A and B generate the dihedral group D_n, that is, every element of D_n can be presented as a product of A and B in some order.

Solution: (a) Since A and B are reflections, we have $A^2 = B^2 = I$. The composition AB is a rotation by $\frac{2\pi}{n}$, so $(AB)^n$ is a rotation by $n \cdot \frac{2\pi}{n} = 2\pi$, so it is the identity map.

(b) The power $(AB)^k$ is a rotation by $k \cdot \frac{2\pi}{n}$, so the group generated by A and B contains all rotations in D_n. To show that we can get all the reflections, we have several possible ways to argue:

Solution 1: Let ℓ_1 and ℓ_2 be the lines of reflection for A and B. Let $\ell_3 \neq \ell_2$ be another line which makes angle $\frac{\pi}{n}$ with ℓ_1, and C the reflection in ℓ_3. Then CA is also a rotation by $\frac{2\pi}{n}$, so $CA = AB$. By multiplying both sides of this equation by A, we get $C = CA^2 = ABA$. Similarly, we can get all reflections in the dihedral group in the form $(AB)^k A$.

Solution 2: Without loss of generality, we can assume that A is a reflection in the x-axis, so $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, while $(AB)^k$ is a rotation by $k \cdot \frac{2\pi}{n}$ and hence given by the matrix

$$(AB)^k = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & -\sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}.$$

Now

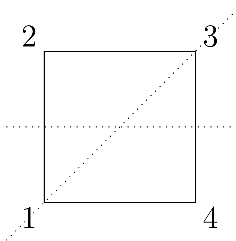
$$(AB)^k A = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & -\sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & -\cos(\frac{2\pi k}{n}) \end{pmatrix}.$$

As k varies, we get all reflections in D_n.

Solution 3: Let H be the subgroup of D_n generated by A and B. It contains at least $n + 2$ elements: n rotations, A and B. On the other hand, by Lagrange Theorem $|H|$ divides $|D_n| = 2n$, so $|H| = 2n$ and $H = D_n$.

2. Consider the subgroup in S_4 generated by (2 4) and (1 2)(3 4). Prove that it contains 8 elements and is isomorphic to D_4.

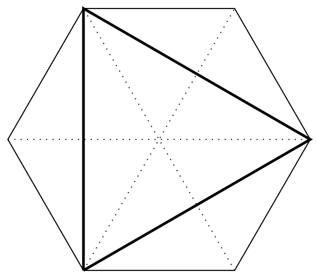
Solution: We label the vertices of the square as follows:



Then (2 4) and (1 2)(3 4) correspond to the reflections in the dotted lines which make angle $\frac{\pi}{4}$. By Problem 1, these reflections generate D_4.

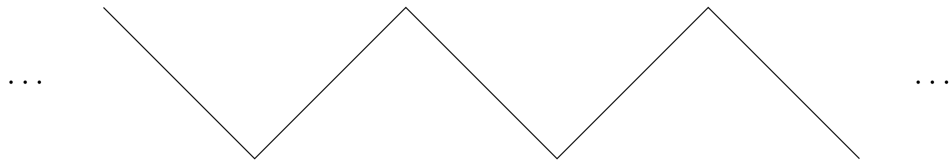
3. Find a subgroup in D_6 isomorphic to D_3.

Solution: Consider the triangle made out of every other vertex of the regular hexagon:

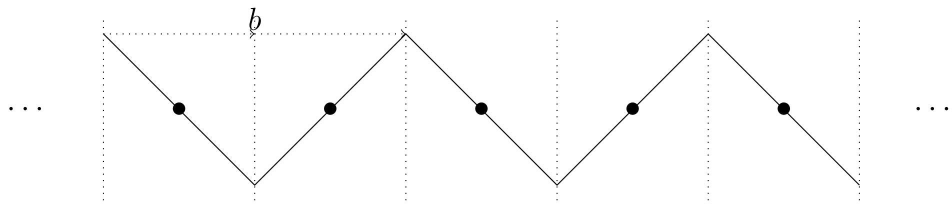


The group of isometries of the hexagon preserving the triangle consists of identity and rotations by $\frac{2\pi}{3} = 2 \cdot \frac{2\pi}{6}$, $\frac{4\pi}{3}$, as well as reflections in dotted lines. It is clearly isomorphic to D_3.

4. Describe all isometries of the infinite pattern:



Solution: This pattern has isometries of several types:



- Rotations by π around points marked by $\bullet$.
- Translations in horizontal direction by kb, where k is an arbitrary integer
- Reflections in the vertical dotted lines
- Glide reflections: reflection in the horizontal line followed by translations by $\frac{b}{2}+kb$ for arbitary integer k.