MAT 150A, Fall 2023
Solutions to homework 2

1. Let x, y, z and w be elements if a group G.
a) Solve for y given that $xyz^{-1}w = 1$.
b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?

Solution: (a) Let us multiply by x^{-1} on the left:

$$x^{-1}xyz^{-1}w = x^{-1},\ yz^{-1}w = x^{-1}.$$

Now multiply by w^{-1} on the right:

$$yz^{-1}ww^{-1} = yz^{-1} = x^{-1}w^{-1},$$

and multiply by z on the right:

$$yz^{-1}z = x^{-1}w^{-1}z.$$

Therefore $y = x^{-1}w^{-1}z$.
(b) Let us solve the equation $xyz = 1$ for z:

$$xyz = 1,\ z = y^{-1}x^{-1}xyz = y^{-1}x^{-1}.$$

Now

$$yzx = y \cdot y^{-1}x^{-1} \cdot x = 1.$$

On the other hand, $yxz \neq 1$ if $xz \neq zx$. For example, pick $x = (1\ 2)$ and $y = (2\ 3)$ in the group S_3, then

$$z = y^{-1}x^{-1} = (2\ 3)(1\ 2) = (1\ 3\ 2),$$

$$yzx = (2\ 3)(1\ 3\ 2)(1\ 2) = e \text{ while } yxz = (2\ 3)(1\ 2)(1\ 3\ 2) = (1\ 2\ 3) \neq e.$$

2. How many elements of order 2 does the symmetric group S_4 contain?
Solution: The order of a permutations is the least common multiple of the lengths of disjoint cycles in it. If the order is equal to 2, this means that all cycles have length 2. In S_4, we have 6 transpositions (cycles of length 2), and 3 products of to disjoint transpositions ($(1\ 2)(3\ 4)$, $(1\ 4)(2\ 3)$ and $(1\ 3)(2\ 4)$). Therefore there are 9 permutations of order 2 in S_4.

3. Find the order of the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 10 & 7 & 6 & 2 & 5 & 9 & 4 & 1 & 8 \end{pmatrix}$$

Solution: We have

$$f = (1\ 3\ 7\ 9)(2\ 10\ 8\ 4\ 6\ 5)$$

The cycles in f have length 4 and 6, so the order of f equals $LCM(4, 6) = 12$.

4. Determine if the following subset H is a subgroup of G:
a) $G = (\mathbb{Z}, +)$ and H is the set of positive integers.
b) $G = (\mathbb{R}^*, \cdot)$ and H is the set of positive real numbers.
c) G is the set of invertible $n \times n$ matrices and H is the set of $n \times n$ matrices with determinant 1.

Solution
(a) No: 1 is in H while (-1) is not in H, so H is not closed under taking inverses.
(b) Yes: $1 > 0$, so H contains the identity. If $a, b > 0$ then $ab > 0$, so H is closed under multiplication. If $a > 0$ then $a^{-1} = \frac{1}{a} > 0$, so H is closed under taking inverses. Therefore H is a subgroup of G.

(c) Yes: $\det(I) = 1$, so H contains the identity matrix. If $\det(A), \det(B) = 1$ then $\det(AB) = \det(A)\det(B) = 1$, so AB is in H. Finally, if $\det(A) = 1$ then A is invertible and $\det(A^{-1}) = 1/\det(A) = 1$, so A^{-1} is in H.