MAT 150A, Fall 2023
Solutions to homework 4

1. (25 points) Let G be a group of order 25. Prove that G has at least one subgroup of order 5, and that if it contains only one subgroup of order 5 then it is a cyclic group.

Solution: Let x be an element of G. By Lagrange Theorem the order of x divides 25, so it could be equal to 1, 5 or 25. We have the following cases:

a) $\mathrm{Ord}(x) = 1$, then $x = e$ is the identity element.

b) $\mathrm{Ord}(x) = 25$, then G is cyclic and generated by x. It has a subgroup $\langle x^5 \rangle = \{1, x^5, x^{10}, x^{15}, x^{20}\}$ of order 5.

c) Order of every non-identity element equals 5. Then each non-identity element generates a subgroup of order 5. If $\{1, x, x^2, x^3, x^4\}$ is one such subgroup and y is not contained in it then y generates another subgroup of order 5. So we have more than one subgroup of order 5.

2. (25 points) Is it possible to construct an injective homomorphism (a) from $\mathbb{Z}_3$ to $\mathbb{Z}_4$? (b) From S_3 to S_4?

Solution: (a) (15 points) (**First solution**) No. Assume that $\varphi : \mathbb{Z}_3 \to \mathbb{Z}_4$ is an injective homomorphism, then $\mathrm{Im}(\varphi)$ is a subgroup $\mathbb{Z}_4$ with 3 elements. By Lagrange Theorem the size of $\mathrm{Im}(\varphi)$ must divide $|\mathbb{Z}_4| = 4$, contradiction.

(**Second solution**) Assume that $\varphi(1) = a$, then $\varphi(3) = \varphi(1 + 1 + 1) = \varphi(1) + \varphi(1) + \varphi(1) = a + a + a = 3a$. But $\varphi(3) = \varphi(0) = 0$, so $3a = 0$ in $\mathbb{Z}_4$. Since 3 and 4 are coprime, we get $a = 0 \mod 4$, contradiction.

(b) (10 points) Yes: given a permutation f in S_3, the permutation $\varphi(f)$ in S_4 is defined by

$$\varphi(f)(i) = \begin{cases} f(i) & i = 1, 2, 3 \\ 4 & i = 4. \end{cases}$$

so that $\varphi(f)$ fixes 4. Clearly, this is an injective homomorphism.

3. (25 points) A finite group G contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?

Solution: By Lagrange Theorem the order of G is divisible by 10 and by 6, so it is divisible by 30.

4. (25 points) Let $\varphi : G_1 \to G_2$ be a group homomorphism. Suppose that $|G_1| = 18, |G_2| = 15$ and that φ is not the trivial homomorphism. What is the order of the kernel?

Solution: By Counting Formula $|\mathrm{Ker}\varphi| \cdot |\mathrm{Im}\varphi| = |G_1| = 18$ and by Lagrange Theorem $|\mathrm{Im}\varphi|$ divides $|G_2| = 15$. Therefore $|\mathrm{Im}\varphi|$ divides both 15 and 18 and it could be equal to 1 or 3. Since φ is nontrivial, we get $|\mathrm{Im}\varphi| = 3$ and $|\mathrm{Ker}\varphi| = 6$.