MAT 150A, Fall 2023
Solutions to homework 6

1. (25 points) Are the following matrices orthogonal? If they are, describe them geometrically.

(a)(15 points) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (b) (10 points) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

Solution: a) We have

$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq I$$

so A is not orthogonal.

b) We have

$$B^T B = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{3}{4} + \frac{1}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & \frac{1}{4} + \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

so B is orthogonal.

2. (25 points) Find all diagonal 3×3 matrices

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

which are orthogonal.

Solution: We have

$$A^T A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

The matrix A is orthogonal if $a^2 = b^2 = c^2 = 1$, so $a, b, c \in \{-1, 1\}$.

Recall that an orthogonal matrix A is called **orientation reversing** if $\det(A) = -1$ and **orientation preserving** if $\det(A) = 1$.

3. (25 points) Find all 2×2 orientation reversing matrices of finite order.

Solution: Any orientation reversing 2×2 orthogonal matrix is a reflection and has order 2.

4. (25 points) Find all 2×2 orientation preserving matrices of finite order.

Solution: Any orientation reversing 2×2 orthogonal matrix is a rotation by some order φ. It has finite order n if $m\varphi = 2\pi n$ and $\varphi = \frac{2\pi n}{m}$ for some integer n.