MAT 150C, Spring 2017
Practice problems for the final exam

This practice sheet contains more problems than the actual exam. Problem marked with stars are more complicated than others.

1. The group S_3 acts on the set of all subsets of $\{1, 2, 3\}$.

a) Compute the character of the corresponding representation of S_3.

b) Decompose it into irreducibles.

2. Consider the square on the plane, let l_1 and l_2 be two lines perpendicular to the sides of the square.

a) Prove that the group G generated by reflections in l_1 and l_2 is isomorphic to $\mathbb{Z}_2 \times Z_2$.

b) Compute the character table of G.

c) Consider the action of G on the set of vertices of the square. Compute the character of the corresponding representation and decompose it into irreducibles.

d) Consider the action of G on the set of sides of the square. Compute the character of the corresponding representation and decompose it into irreducibles.

e) Consider the action of G on the set of diagonals of the square. Compute the character of the corresponding representation and decompose it into irreducibles.

3. Compute the character table for the dihedral group D_4.

4. a) Let A be an $n \times n$ matrix with rational coefficients. Prove that all eigenvalues of A are algebraic over $\mathbb{Q}$.

b) Give an example of a 3×3 matrix with rational coefficients such that its eigenvalues are not rational.

c) Suppose that none of eigenvalues of a 3×3 matrix A with rational coefficients is rational. Prove that A is diagonalizable.

d)** Prove that every algebraic number of degree n is an eigenvalue of some $n \times n$ matrix with rational coefficients.

5. A complex number $\alpha = x + iy$ is algebraic over $\mathbb{Q}$.

a) Prove that $\overline{\alpha} = x - iy$ is algebraic.

b) Prove that x and y are algebraic.

6. a) Compute the minimal polynomial $m(x)$ for the algebraic number $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

b) Find the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

c) Prove that $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\alpha)$

d) Prove that $\mathbb{Q}(\alpha)$ is the splitting field for $m(x)$.

e)** Compute the Galois group of $m(x)$.

7. a) Prove that the polynomial $x^2 - 2$ is irreducible over $\mathbb{Z}_3$.

b) Let $F = \mathbb{Z}_3(\alpha)$ where $\alpha^2 = 2$. Compute $(1 + \alpha)^{10}$ in F.

c) Compute $1/(1 + 2\alpha)$ in F.

d) Compute the Galois group of F over $\mathbb{Z}_3$.

e) Prove that the polynomial $x^2 + x + 2$ has two roots in F, but does not have a root in $\mathbb{Z}_3$.

8. How many non-isomorphic (not necessary irreducible) representations of $SU(2)$ of dimension (a) 3 (b) 4 are there?

9. How many non-isomorphic (not necessary irreducible) representations of S_3 of dimension 3 are there?

10. Three points A, B, C belong to the same circle with center at O. Suppose the coordinates of A, B, C are constructible numbers. Prove that the coordinates of O and the radius of the circle are also constructible.

11. Prove that $\{I, -I\}$ is a normal subgroup in $SU(2)$.

12. Let A_5 denote the group of even permutations on 5 letters. Prove that every homomorphism $A_5 \to G$ is either trivial or injective.