MAT 17B, Fall 2020
Solutions to homework 4

1. (10 points) Find the area of the region bounded by the curves $y = x^2 - 2x$ and $y = x + 4$.

Solution: First, let us find the intersection points of these two curves:

$$x^2 - 2x = x + 4 \Leftrightarrow x^2 - 3x - 4 = 0 \Leftrightarrow (x+1)(x-4) = 0,$$

so $x = -1$ or $x = 4$. Next, we compute the integral:

$$\int_{-1}^{4} (f(x) - g(x))dx = \int_{-1}^{4} ((x^2 - 2x) - (x+4))dx =$$

$$\int_{-1}^{4} (x^2 - 3x - 4)dx = \left[\frac{x^3}{3} - 3\frac{x^2}{2} - 4x\right]_{-1}^{4} =$$

$$[\frac{64}{3} - \frac{3 \cdot 16}{2} - 16] - [\frac{-1}{3} - \frac{3}{2} + 4] = \frac{65}{3} - \frac{45}{2} - 20 =$$

$$\frac{130}{6} - \frac{135}{6} - \frac{120}{6} = -\frac{125}{6}.$$

Since this integral is negative, we conclude that $f(x) < g(x)$ on $[-1, 4]$ and the area equals $\frac{125}{6}$.

2. (10 points) Find the volume of the solid obtained by rotating the region bounded by curves $y = x^2 - 2x$ and $y = x + 4$ about the x-axis.

Solution 1: We follow the computation from problem 1: the curves intersect at $x = -1$ and $x = 4$, and $g(x) > f(x)$ on $[-1, 4]$. Therefore the volume equals

$$\int_{-1}^{4} \pi(g^2(x) - f^2(x))dx = \pi \int_{-1}^{4} ((x+4)^2 - (x^2 - 2x)^2)dx =$$

$$\pi \int_{-1}^{4} (x^2 + 8x + 16 - x^4 + 4x^3 - 4x^2)dx =$$

$$\pi \int_{-1}^{4} (-x^4 + 4x^3 - 3x^2 + 8x + 16)dx = \pi \left[-\frac{x^5}{5} + x^4 - x^3 + 4x^2 + 16x\right]_{-1}^{4} =$$

$$\pi[-\frac{1024}{5} + 256 - 64 + 64 + 64] - \pi[\frac{1}{5} + 1 + 1 + 4 - 16] = \pi[-\frac{1025}{5} + 330] =$$

$$\pi[-205 + 330] = 125\pi.$$

Solution 2: Note that the part of the graph of $f(x) = x^2 - 2x$ on the interval $[0, 2]$ is below the x-axis, and we need to ignore it to avoid double-counting the volume. Then the volume is given by the sum of three integrals:

$$\pi\int_{-1}^{0}(g^2(x)-f^2(x))dx+\pi\int_{0}^{2}g^2dx+\pi\int_{2}^{4}(g^2(x)-f^2(x))dx.$$

As above, $g^2(x)-f^2(x)=-x^4+4x^3-3x^2+8x+16$ and

$$\int(g^2(x)-f^2(x))dx=-\frac{x^5}{5}+x^4-x^3+4x^2+16x+C$$

On the other hand,

$$\int g^2(x)dx=\int(x+4)^2dx=\frac{1}{3}(x+4)^3+C$$

Therefore the volume is given by the formula

$$\pi\left[-\frac{x^5}{5}+x^4-x^3+4x^2+16x\right]_{-1}^{0}+\pi\left[\frac{1}{3}(x+4)^3\right]_{0}^{2}+$$

$$\pi\left[-\frac{x^5}{5}+x^4-x^3+4x^2+16x\right]_{2}^{4}=$$

$$0-\pi[\frac{1}{5}+1+1+4-16]+\frac{216}{3}-\frac{64}{3}+$$

$$\pi[-\frac{1024}{5}+256-64+64+64]-\pi[-\frac{32}{5}+16-8+16+32]=$$

$$\pi[-\frac{993}{5}+274+\frac{152}{3}]=\pi[-\frac{2979}{15}+\frac{4110}{15}+\frac{760}{15}]=\frac{1891}{15}\pi$$

Comment: Although the second solution is the only one mathematically correct, it is more complicated than the first one and this was not intended. The two answers differ by $\pi\int_0^2(x^2-2x)^2dx=\frac{16}{15}\pi$ (check it!). Nevertheless, the first solution would receive a full credit if done correctly.

3. (10 points) Find the volume of the solid obtained by rotating the region bounded by curves $y=e^x$, $x=1$, $x=3$ and $y=0$ about the x-axis.

Solution: The volume equals

$$\pi\int_1^3(e^x)^2dx=\pi\int_1^3e^{2x}dx=\frac{\pi}{2}e^{2x}|_1^3=\frac{\pi}{2}(e^6-e^2).$$