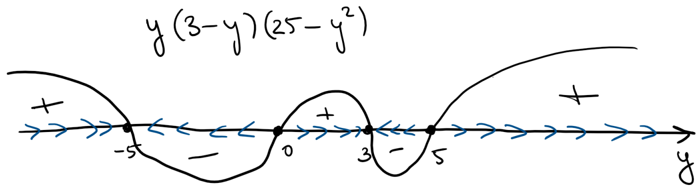
MAT 17B, Fall 2020
Homework 5

1. (10 points) Sketch the phase plot for the autonomous differential equation $y' = y(3-y)(25-y^2)$. Find all equilibria and determine if each is locally stable or unstable.

Solution: We have $y(y-3)(25-y^2) = y(y-3)(5-y)(5+y)$. The equilibrium solutions correspond to $y' = 0$, so these are $y = -5, y = 0, y = 3$ and $y = 5$. The function $y(y-3)(5-y)(5+y)$ is positive for $y > 5$ and changes the sign at each equilibrium, so the phase plot has the following shape:



For the phase plot, we see that $y = -5$ ad $y = 3$ are stable, while $y = 0$ and $y = 5$ are unstable.

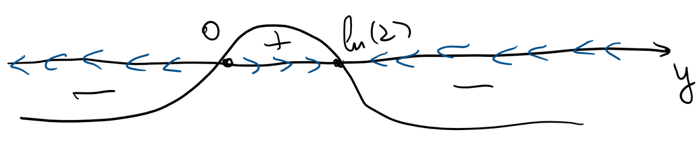
2. (10 points) Sketch the phase plot for the autonomous differential equation $y' = 5y(2e^{-y} - 1)$. Find all equilibria and determine if each is locally stable or unstable.

Solution: The equilibrium solutions correspond to $y' = 0$, so either $y = 0$ or $2e^{-y} = 0 \Leftrightarrow e^y = 2 \Leftrightarrow y = \ln(2)$.

If $y > \ln(2)$ then $e^y > 2$, so $e^{-y} < 1/2$ and $2e^{-y} - 1 < 0$, so $y' < 0$.

If $0 < y < \ln(2)$ then $e^y < 2$, so $e^{-y} > 1/2$ and $2e^{-y} - 1 > 0$, so $y' > 0$.

If $y < 0$ then similarly $2e^{-y} - 1 > 0$, so $y' < 0$. The phase plot has the following shape:



From the phase plot, we see that $y = 0$ is unstable and $y = \ln(2)$ is stable.

3. (10 points) The population size $N(t)$ satisfies the differential equation

$$N' = 2(1 - \frac{N}{1000})N - hN,$$

where h is a parameter called *harvest rate*. Assume $h \neq 2$.

a) Find all equilibria for this equation (the answer depends on h).

b) Determine if these equilibria are stable or unstable (the answer depends on h).

c) Find the limit $\lim_{t\to\infty} N(t)$ depending on the initial condition $N(0) > 0$ (the answer depends on h).

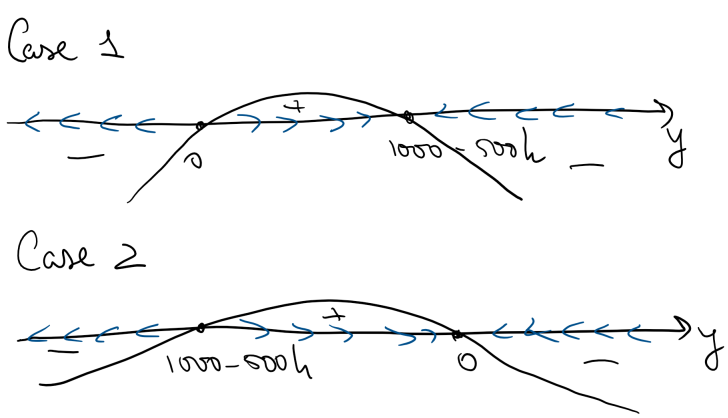
Solution: First, we can factor N out of the right hand side:

$$N' = \left[2(1 - \frac{N}{1000}) - h\right] N.$$

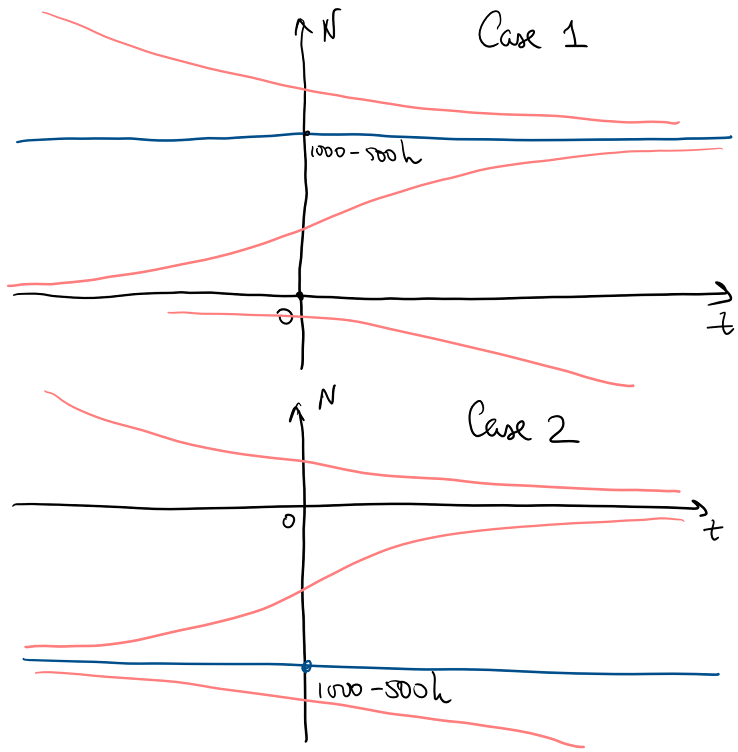
The equilibria are given by the equations $N = 0$, and

$$2(1 - \frac{N}{1000}) - h = 0,\ 2 - \frac{N}{500} - h = 0,\ \frac{N}{500} = 2 - h,\ N = 1000 - 500h.$$

There are two cases depending on h: either $1000 - 500h > 0$ (equivalently, $h < 2$), or $1000 - 500h < 0$ (equivalently, $h > 2$). The corresponding phase plots are the following:



In Case 1 $N = 0$ is unstable while $N = 1000 - 500h$ is stable. In Case 2 $N = 0$ is stable while $N = 1000 - 500h$ is unstable. The graphs of solutions $N(t)$ have the following form:



In Case 1 we have $\lim_{t\to+\infty} N(t) = 1000 - 500h$ both if $N(0) > 1000 - 500h$ and if $0 < N(0) < 1000 - 500h$. In Case 2 we have $\lim_{t\to+\infty} N(t) = 0$ for $N(0) > 0$. We can sum up this as follows:

$$\lim_{t\to+\infty} N(t) = \begin{cases} 1000 - 500h & \text{if } N(0) > 0 \text{ and } 1000 - 500h > 0 \text{ (equivalently, } h < 2\text{)}, \\ 0 & \text{if } N(0) > 0 \text{ and } 1000 - 500h < 0 \text{ (equivalently, } h > 2\text{)}. \end{cases}$$