MAT 17B, Fall 2020
Solutions to homework 8

1. (10 points) Consider the matrices $A = \begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. Compute the products AB and BA.

Solution: We have

$$AB = \begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot 0 \\ -7 \cdot 2 + 2 \cdot 1 & -7 \cdot 1 + 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -12 & -7 \end{pmatrix}$$

and

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot (-7) & 2 \cdot 3 + 1 \cdot 2 \\ 1 \cdot 1 + 0 \cdot (-7) & 1 \cdot 3 + 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} -5 & 8 \\ 1 & 3 \end{pmatrix}$$

2. (10 points) Solve the linear system of equations

$$\begin{cases} 2x_1 - x_2 = 7 \\ 3x_1 + 2x_2 = 5 \end{cases}$$

Solution 1: If we multiply the first equation by 2, we get $4x_1 - 2x_2 = 14$, add it to the second equation and get $7x_1 = 19$, so $x_1 = \frac{19}{7}$. Now from the first equation $x_2 = 2x_1 - 7 = \frac{38}{7} - 7 = -\frac{11}{7}$.

Solution 2: We have

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{2 \cdot 2 - (-1) \cdot 3} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{pmatrix},$$

so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} \cdot 7 + \frac{1}{7} \cdot 5 \\ -\frac{3}{7} \cdot 7 + \frac{2}{7} \cdot 5 \end{pmatrix} = \begin{pmatrix} \frac{19}{7} \\ \frac{-11}{7} \end{pmatrix}.$$

3. (10 points) Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Solution: The characteristic equation has the form

$$\det \begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - (-2) \cdot 1 = 4 - \lambda - 4\lambda + \lambda^2 + 2 =$$

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0.$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

The eigenvector for $\lambda_1 = 2$ satisfies the equation

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $-x+y=0, -2x+2y=0$ and $x=y$. We can choose $(1,1)$ or any multiple of it as an eigenvector.

The eigenvector for $\lambda_2 = 3$ satisfies the equation

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $-2x+y=0$, and $y=2x$. We can choose $(1,2)$ or any multiple of it as an eigenvector.