MAT 17C, Fall 2017

Answers for practice problems for Midterm 2

1. a) Critical point $(0,0)$, saddle
b) No critical points
c) Critical point $(3/4,-3/4)$, minimum
d) 4 critical points: minimum at $(1,1)$, saddles at $(1,-1)$ and $(-1,1)$, maximum at $(-1,-1)$.

2. a) $\lambda_1 = 0,\ v_1 = (1,-1),\ \lambda_2 = 2,\ v_2 = (1,1)$;
b) $\lambda_1 = 1,\ v_1 = (1,1),\ \lambda_2 = 2,\ v_2 = (1,2)$;
c) $\lambda_1 = -2,\ v_1 = (1,-1),\ \lambda_2 = 2,\ v_2 = (3,1)$;

3. a) $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$
b) $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$
c) $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$
d) $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

4. a) $\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3}{2} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$
b) $\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2} e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \frac{1}{2} e^{4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

5*. The characteristic polynomial has the form

$$(a-\lambda)(c-\lambda) - b^2 = ac - a\lambda - c\lambda + \lambda^2 - b^2 = \lambda^2 - (a+c)\lambda + (ac - b^2).$$

We need to prove that

$$(a+c)^2 - 4(ac - b^2) \geq 0.$$

Indeed,

$$(a+c)^2 - 4(ac-b^2) = a^2 + 2ac + c^2 - 4ac + 4b^2 = a^2 - 2ac + c^2 + 4b^2 = (a-c)^2 + b^2 \geq 0.$$

6. a) $\lambda_{1,2} = \pm\sqrt{3}i$, center, stable.
b) $\lambda_{1,2} = 2, 3$, unstable node.
c) $\lambda_{1,2} = -1, 6$, saddle, unstable.
d) $\lambda_{1,2} = 1 \pm 2i$, unstable spiral.