MAT 17C, Fall 2017

Practice problems for the final exam

This practice sheet has more problems than the actual exam

1. Prove that the limit

$$\lim_{(x,y)\to(0,)} \frac{x^2 - 5y^3}{x^2 + 3y^3}$$

does not exist.

2. Compute the partial derivatives of the functions:

a) $x^2 - y^2$

b) $\frac{\sin(x+y)}{\cos(x-y)}$

c) $\ln(x + e^y)$.

3. Find the gradient of the function $\ln(x + e^y)$ at the point $(1, 0)$.

4. Use implicit differentiation to find the slope of the tangent line to the curve $x^3 + y^3 = 28$ at the point $(1, 3)$.

5. Find the directional derivative of the function $f(x, y) = e^x e^y - xy$ at the point $(1, 2)$ in direction $u = (1, 1)$.

6. Find all the critical points of a given function and determine their type (local minimum, local maximum, saddle):

a) $f(x, y) = x\ln(x) + y^2$

b) $f(x, y) = x^2 + xy + y^2$

c) $f(x, y) = e^{-x^2-y^2}$.

7. Find the equation of the tangent plane to the graph of the function

$$f(x, y) = x^3 + ye^x$$

at the point $(1, 0)$.

8. Solve the system of linear differential equations (if the eigenvalues are real) and sketch the phase portrait:

a)

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

b)

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

9. Solve the second order linear differential equations:

a) $x''(t) + 3x'(t) + 2x(t) = 0$

b) $x''(t) = x(t)$

c) $x"(t) - x'(t) = 0,\ x(0) = 1,\ x'(0) = 2.$

10. Solve the initial value problem:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ x(0) = 2,\ y(0) = 0.$$

11. Let $N(t)$ denote the population of rabbits and $P(t)$ denote the population of wolves. This predator-prey system is described by a system of differential equations:

$$\frac{dN}{dt} = 3N - PN,\ \frac{dP}{dt} = -2P + PN.$$

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

12. Let $N_1(t)$ and $N_2(t)$ denote the populations of two species competing for limited resources. They are described by a system of differential equations:

$$\frac{dN_1}{dt} = N_1(1 - \frac{N_1}{35} - 3\frac{N_2}{35}),$$

$$\frac{dN_2}{dt} = 3N_2(1 - \frac{N_2}{40} - 4\frac{N_1}{40}),$$

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

13. In a model for the forest growth, let $x_1(t)$ and $x_2(t)$ denote the areas occupied by the gaps and trees respectively. They are described by a compartment model:

$$\frac{dx_1}{dt} = -0.2x_1 + 0.1x_2,\ \frac{dx_2}{dt} = 0.2x_1 - 0.1x_2.$$

Find a general solution for this system.

14. Consider the system of differential equations:

$$x'(t) = y(t),\ y'(t) = -x(t).$$

a) Show that

$$x(t) = C_1 \sin t + C_2 \cos t, y(t) = C_1 \cos t - C_2 \sin t$$

is a solution of this system for arbitrary constants C_1, C_2.

b) Let $F(t) = x(t)^2 + y(t)^2$. Compute $F'(t)$ in terms of $x(t)$ and $y(t)$.

c) Solve the system with the initial conditions $x(0) = 3, y(0) = 5$.