MAT 17C, Fall 2017

Answers to practice problems for the final exam

1. Prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-5y^3}{x^2+3y^3}$$

does not exist.

Answer: If $y=0$ then

$$\lim_{x\to 0} \frac{x^2-5y^3}{x^2+3y^3} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

If $x=0$ then

$$\lim_{y\to 0} \frac{x^2-5y^3}{x^2+3y^3} = \lim_{y\to 0} \frac{-5y^3}{3y^3} = -5/3.$$

Since the limits along the axis are different, the limit does not exist.

2. Compute the partial derivatives of the functions:

a) x^2-y^2

Answer: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -2y$.

b) $\frac{\sin(x+y)}{\cos(x-y)}$

Answer:

$$\frac{\partial f}{\partial x} = \frac{\cos(x+y)\cos(x-y)+\sin(x+y)\sin(x-y)}{\cos^2(x-y)} = \frac{\cos(2y)}{\cos^2(x-y)}$$

$$\frac{\partial f}{\partial y} = \frac{\cos(x+y)\cos(x-y)-\sin(x+y)\sin(x-y)}{\cos^2(x-y)} = \frac{\cos(2x)}{\cos^2(x-y)}.$$

c) $\ln(x+e^y)$.

Answer: $\frac{\partial f}{\partial x} = \frac{1}{x+e^y}$, $\frac{\partial f}{\partial y} = \frac{e^y}{x+e^y}$.

3. Find the gradient of the function $\ln(x+e^y)$ at the point $(1,0)$.

Answer: $(1/2,\ 1/2)$.

4. Use implicit differentiation to find the slope of the tangent line to the curve $x^3+y^3=28$ at the point $(1,3)$.

Answer:
$$y'(x) = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{3x^2}{3y^2} = -\frac{x^2}{y^2}.$$
At $(x, y) = (1, 3)$ we get $y'(x) = -1/9$.

5. Find the directional derivative of the function $f(x, y) = e^x e^y - xy$ at the point $(1, 2)$ in direction $u = (1, 1)$.

Answer: We have $f_x = e^x e^y - y, f_y = e^x e^y - x$, so
$$D_{(1,1)}f = f_x + f_y = e^x e^y - y + e^x e^y - x = e^1 e^2 - 2 - e^1 e^2 - 1 = 2e^3 - 3.$$

6. Find all the critical points of a given function and determine their type (local minimum, local maximum, saddle):

a) $f(x, y) = x \ln(x) + y^2$

Answer: $(e^{-1}, 0)$, local minimum.

b) $f(x, y) = x^2 + xy + y^2$

Answer: $(0, 0)$, local minimum.

c) $f(x, y) = e^{-x^2-y^2}$.

Answer: $(0, 0)$, local maximum.

7. Find the equation of the tangent plane to the graph of the function
$$f(x, y) = x^3 + ye^x$$
at the point $(1, 0)$.

Answer: $f_x = 3x^2 + ye^x, f_y = e^x$, so $f_x(1, 0) = 3, f_y(1, 0) = e$, and $f(1, 0) = 1$, so the equation of the tangent plane has the form $z - 1 = 3(x - 1) + e(y - 0)$.

8. Solve the system of linear differential equations (if the eigenvalues are real) and sketch the phase portrait:

a)
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Answer:
$$C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

this is a saddle

b)

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Answer: The eigenvalues are equal to $\lambda_{1,2} = 1 \pm 2i$, so this is an unstable spiral.

9. Solve the second order linear differential equations:

a) $x''(t) + 3x'(t) + 2x(t) = 0$

Answer: $x(t) = C_1 e^{-t} + C_2 e^{-2t}$.

b) $x''(t) = x(t)$

Answer: $x(t) = C_1 e^{-t} + C_2 e^{t}$.

c) $x"(t) - x'(t) = 0,\ x(0) = 1,\ x'(0) = 2.$

Answer: $x(t) = -1 + 2e^{t}$.

10. Solve the initial value problem:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ x(0) = 2,\ y(0) = 0.$$

Answer:

$$\frac{1}{2} e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

11. Let $N(t)$ denote the population of rabbits and $P(t)$ denote the population of wolves. This predator-prey system is described by a system of differential equations:

$$\frac{dN}{dt} = 3N - PN,\ \frac{dP}{dt} = -2P + PN.$$

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

Answer: $(0,0)$, saddle, unstable; $(2,3)$, center, stable.

12. Let $N_1(t)$ and $N_2(t)$ denote the populations of two species competing for limited resources. They are described by a system of differential equations:

$$\frac{dN_1}{dt} = N_1(1 - \frac{N_1}{35} - 3\frac{N_2}{35}),$$

$$\frac{dN_2}{dt} = 3N_2(1 - \frac{N_2}{40} - 4\frac{N_1}{40}),$$

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

Answer: Four equilibrium points:
(a) $(0, 0)$, source, unstable.
(b) $(35, 0)$, sink, stable.
(c) $(0, 40)$, sink, stable.
(d) $(85/11, 100/11)$, saddle, unstable.

13. In a model for the forest growth, let $x_1(t)$ and $x_2(t)$ denote the areas occupied by the gaps and trees respectively. They are described by a compartment model:

$$\frac{dx_1}{dt} = -0.2x_1 + 0.1x_2, \ \frac{dx_2}{dt} = 0.2x_1 - 0.1x_2.$$

Find a general solution for this system.

Answer

$$C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-0.3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

14. Consider the system of differential equations:

$$x'(t) = y(t), \ y'(t) = -x(t).$$

a) Show that

$$x(t) = C_1 \sin t + C_2 \cos t, y(t) = C_1 \cos t - C_2 \sin t$$

is a solution of this system for arbitrary constants C_1, C_2.

Solution: Indeed, $x'(t) = C_1 \cos t - C_2 \sin t = y(t), y'(t) = -C_1 \sin t - C_2 \cos t = -x(t)$.

b) Let $F(t) = x(t)^2 + y(t)^2$. Compute $F'(t)$ in terms of $x(t)$ and $y(t)$.

Solution: By the Chain Rule, we have $F'(t) = 2x(t)x'(t) + 2y(t)y'(t) = 2x(t)y(t) + 2y(t)(-x(t)) = 0$.

c) Solve the system with the initial conditions $x(0) = 3, y(0) = 5$.

Solution: We have $x(0) = C_2 = 3, y(0) = C_1 = 5$, so

$$x(t) = 5 \sin t + 3 \cos t, \ y(t) = 5 \cos t - 3 \sin t.$$