1. Prove that the limit
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - 5y^3}{x^2 + 3y^3}
\]
does not exist.

**Answer:** If \(y = 0\) then
\[
\lim_{x \to 0} \frac{x^2 - 5y^3}{x^2 + 3y^3} = \lim_{x \to 0} \frac{x^2}{x^2} = 1
\]
If \(x = 0\) then
\[
\lim_{y \to 0} \frac{x^2 - 5y^3}{x^2 + 3y^3} = \lim_{y \to 0} \frac{-5y^3}{3y^3} = -\frac{5}{3}.
\]
Since the limits along the axis are different, the limit does not exist.

2. Compute the partial derivatives of the functions:

a) \(x^2 - y^2\)

**Answer:** \(\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2y\).

b) \(\frac{\sin(x+y)}{\cos(x-y)}\)

**Answer:**
\[
\frac{\partial f}{\partial x} = \frac{\cos(x+y)\cos(x-y) + \sin(x+y)\sin(x-y)}{\cos^2(x-y)} = \frac{\cos(2y)}{\cos^2(x-y)}
\]
\[
\frac{\partial f}{\partial y} = \frac{\cos(x+y)\cos(x-y) - \sin(x+y)\sin(x-y)}{\cos^2(x-y)} = \frac{\cos(2x)}{\cos^2(x-y)}.
\]

c) \(\ln(x + e^y)\).

**Answer:** \(\frac{\partial f}{\partial x} = \frac{1}{x + e^y}, \frac{\partial f}{\partial y} = \frac{e^y}{x + e^y}\).

3. Find the gradient of the function \(\ln(x + e^y)\) at the point \((1,0)\).

**Answer:** \((1/2, 1/2)\).

4. Use implicit differentiation to find the slope of the tangent line to the curve \(x^3 + y^3 = 28\) at the point \((1,3)\).

**Answer:** \((1/2, 1/2)\).
Answer:
\[
y'(x) = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{3x^2}{3y^2} = -\frac{x^2}{y^2}.
\]
At \((x, y) = (1, 3)\) we get \(y'(x) = -1/9\).

5. Find the directional derivative of the function \(f(x, y) = e^x e^y - xy\) at the point \((1, 2)\) in direction \(u = (1, 1)\).

Answer: We have \(f_x = e^x e^y - y, f_y = e^x e^y - x,\) so
\[
D_{(1,1)}f = f_x + f_y = e^x e^y - y + e^x e^y - x = e^1 e^2 - 2 - e^1 e^2 - 1 = 2e^3 - 3.
\]

6. Find all the critical points of a given function and determine their type (local minimum, local maximum, saddle):

a) \(f(x, y) = x \ln(x) + y^2\)

Answer: \((e^{-1}, 0),\) local minimum.

b) \(f(x, y) = x^2 + xy + y^2\)

Answer: \((0, 0),\) local minimum.

c) \(f(x, y) = e^{-x^2-y^2}\).

Answer: \((0, 0),\) local maximum.

7. Find the equation of the tangent plane to the graph of the function
\[
f(x, y) = x^3 + ye^x
\]
at the point \((1, 0)\).

Answer: \(f_x = 3x^2 + ye^x, f_y = e^x,\) so \(f_x(1, 0) = 3, f_y(1, 0) = e,\) and \(f(1, 0) = 1,\) so the equation of the tangent plane has the form \(z - 1 = 3(x - 1) + e(y - 0).\)

8. Solve the system of linear differential equations (if the eigenvalues are real) and sketch the phase portrait:

a) \[
\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}
\]

Answer:
\[
C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
this is a saddle

b) 

\[
\begin{pmatrix}
x'(t) \\
y'(t)
\end{pmatrix} =
\begin{pmatrix}
0 & 5 \\
-1 & 2
\end{pmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
\]

**Answer:** The eigenvalues are equal to \( \lambda_{1,2} = 1 \pm 2i \), so this is an unstable spiral.

9. Solve the second order linear differential equations:

a) \( x''(t) + 3x'(t) + 2x(t) = 0 \)

**Answer:** \( x(t) = C_1e^{-t} + C_2e^{-2t} \).

b) \( x''(t) = x(t) \)

**Answer:** \( x(t) = C_1e^{-t} + C_2e^{t} \).

c) \( x''(t) - x'(t) = 0, \ x(0) = 1, \ x'(0) = 2 \).

**Answer:** \( x(t) = -1 + 2e^{t} \).

10. Solve the initial value problem:

\[
\begin{pmatrix}
x'(t) \\
y'(t)
\end{pmatrix} =
\begin{pmatrix}
0 & 3 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}, \ x(0) = 2, \ y(0) = 0.
\]

**Answer:**

\[
\frac{1}{2}e^{-t}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

11. Let \( N(t) \) denote the population of rabbits and \( P(t) \) denote the population of wolves. This predator-prey system is described by a system of differential equations:

\[
\frac{dN}{dt} = 3N - PN, \quad \frac{dP}{dt} = -2P + PN.
\]

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

**Answer:** \((0,0)\), saddle, unstable; \((2,3)\), center, stable.

12. Let \( N_1(t) \) and \( N_2(t) \) denote the populations of two species competing for limited resources. They are described by a system of differential equations:

\[
\frac{dN_1}{dt} = N_1(1 - \frac{N_1}{35} - 3\frac{N_2}{35}),
\]

\[
\frac{dN_2}{dt} = N_2(1 - \frac{N_2}{35} - 3\frac{N_1}{35}).
\]
\[ \frac{dN_2}{dt} = 3N_2(1 - \frac{N_2}{40} - \frac{4N_1}{40}), \]

(a) Find all equilibrium points for this system; (b) Determine the type and stability of each equilibrium; (c) Sketch the phase portrait of the system.

**Answer:**

(a) (0, 0), source, unstable.
(b) (35, 0), sink, stable.
(c) (0, 40), sink, stable.
(d) \( \left( \frac{85}{11}, \frac{100}{11} \right) \), saddle, unstable.

13. In a model for the forest growth, let \( x_1(t) \) and \( x_2(t) \) denote the areas occupied by the gaps and trees respectively. They are described by a compartment model:

\[ \frac{dx_1}{dt} = -0.2x_1 + 0.1x_2, \quad \frac{dx_2}{dt} = 0.2x_1 - 0.1x_2. \]

Find a general solution for this system.

**Answer**

\[ C_1 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + C_2 e^{-0.3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]

14. Consider the system of differential equations:

\[ x'(t) = y(t), \quad y'(t) = -x(t). \]

(a) Show that

\[ x(t) = C_1 \sin t + C_2 \cos t, \quad y(t) = C_1 \cos t - C_2 \sin t \]

is a solution of this system for arbitrary constants \( C_1, C_2 \).

**Solution:**

Indeed, \( x'(t) = C_1 \cos t - C_2 \sin t = y(t) \), \( y'(t) = -C_1 \sin t - C_2 \cos t = -x(t) \).

(b) Let \( F(t) = x(t)^2 + y(t)^2 \). Compute \( F'(t) \) in terms of \( x(t) \) and \( y(t) \).

**Solution:**

By the Chain Rule, we have

\[ F'(t) = 2x(t)x'(t) + 2y(t)y'(t) = 2x(t)y(t) + 2y(t)(-x(t)) = 0. \]

(c) Solve the system with the initial conditions \( x(0) = 3, y(0) = 5 \).

**Solution:**

We have \( x(0) = C_2 = 3, y(0) = C_1 = 5 \), so

\[ x(t) = 5 \sin t + 3 \cos t, \quad y(t) = 5 \cos t - 3 \sin t. \]