MAT 17C, Fall 2017
Solutions to homework 5

Section 11.1 Find the general solution of each given system of differential equations and sketch the lines in the direction of the eigenvectors. Indicate on each line the direction in which the solution would move if it starts on that line.

14. (25 points) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: First, we find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$. We have:

$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) - 4 =$$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 = \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3),$$

so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Now

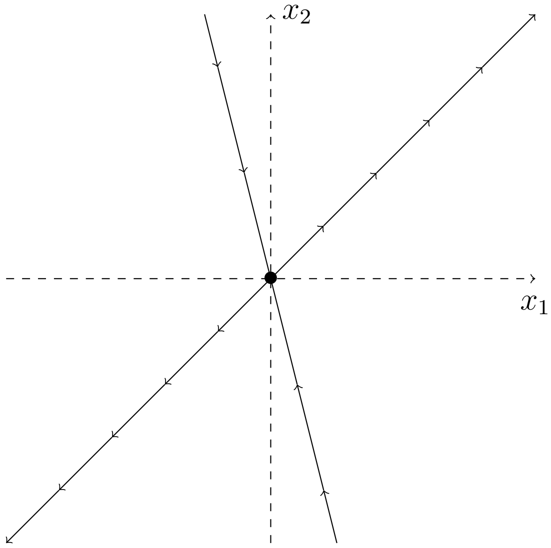
$$\begin{pmatrix} 2-(-2) & 1 \\ 4 & -1-(-2) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix},$$

so the coefficients of the first eigenvector satisfy $4x + y = 0$, and we can choose $x = 1, y = -4$. Similarly,

$$\begin{pmatrix} 2-3 & 1 \\ 4 & -1-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix},$$

so the coefficients of the second eigenvector satisfy $-x + y = 0, 4x - 4y = 0$, and we can choose $x = y = 1$. Therefore the general solution has the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



18. (25 points) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: First, we find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 5 & 2 \\ 1 & 6 \end{pmatrix}$. We have:

$$0 = \det \begin{pmatrix} 5-\lambda & 2 \\ 1 & 6-\lambda \end{pmatrix} = (5-\lambda)(6-\lambda) - 2 =$$

$$30 - 5\lambda - 6\lambda + \lambda^2 - 2 = \lambda^2 - 11\lambda + 28 = (\lambda - 4)(\lambda - 7),$$

so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 7$. Now

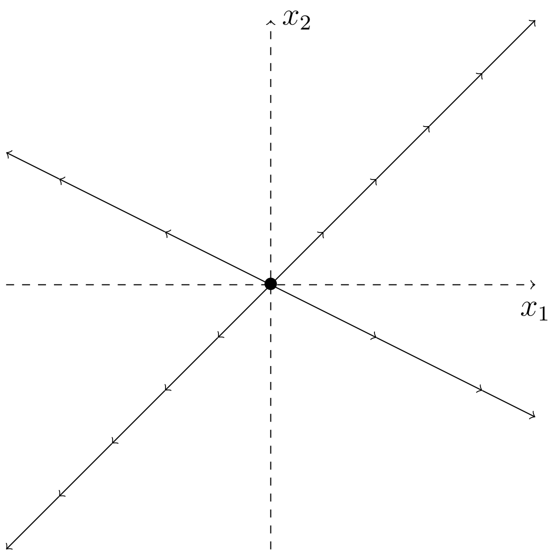
$$\begin{pmatrix} 5-4 & 2 \\ 1 & 6-4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

so the coefficients of the first eigenvector satisfy $x + 2y = 0$, and we can choose $x = 2, y = -1$. Similarly,

$$\begin{pmatrix} 5-7 & 2 \\ 1 & 6-7 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix},$$

so the coefficients of the second eigenvector satisfy $x - y = 0, -2x + 2y = 0$, and we can choose $x = y = 1$. Therefore the general solution has the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



22. (25 points) Solve the initial-value problem:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; x_1(0) = -1, \; x_2(0) = -2.$$

Solution: First, we find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$. We have:

$$0 = \det \begin{pmatrix} -1-\lambda & 0 \\ 1 & -2-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda),$$

so the eigenvalues are equal to $\lambda_1 = -1, \lambda_2 = -2$. Now

$$\begin{pmatrix} -1-(-1) & 0 \\ 1 & -2-(-1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

so the coefficients of the first eigenvector satisfy $x - y = 0$, and we can choose $x = y = 1$. Similarly,

$$\begin{pmatrix} -1-(-2) & 0 \\ 1 & -2-(-2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

so the coefficients of the second eigenvector satisfy $x = 0$, and we can choose $y = 1$. Therefore the general solution has the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At $t = 0$ we get

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so

$$C_1 = -1, \ C_1 + C_2 = -2 \ \Rightarrow \ C_2 = -1.$$

Finally,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

32. (25 points) Consider the differential equation

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Analyze the stability of the equilibrium $(0, 0)$, and classify the equilibrium according to whether it is a sink, a source, or a saddle point.

Solution: First, we find the eigenvalues of the matrix $A = \begin{pmatrix} -5 & -2 \\ 6 & 3 \end{pmatrix}$. We have:

$$0 = \det \begin{pmatrix} -5-\lambda & -2 \\ 6 & 3-\lambda \end{pmatrix} = (-5-\lambda)(3-\lambda) + 12 =$$

$$-15 + 5\lambda - 3\lambda + \lambda^2 + 12 = \lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3).$$

so the eigenvalues are equal to $\lambda_1 = 1, \lambda_2 = -3$. Since they have different signs, this is a saddle, and the equilibrium is unstable.