MAT 17C, Fall 2017
Solutions to homework 6

Section 11.3 Analyze the stability of the equilibrium point $(0,0)$:
2. (25 points)

$$\frac{dx_1}{dt} = -x_1 - x_2 + x_1^2, \frac{dx_2}{dt} = x_2 - x_1^2.$$

Solution: The Jacobi matrix has the form:

$$J = \begin{pmatrix} -1+2x_1 & -1 \\ -2x_1 & 1 \end{pmatrix},$$

so

$$J(0,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

This is a triangular matrix with eigenvalues $1, -1$, so this is a saddle point and the equilibrium is unstable.

4. (25 points)

$$\frac{dx_1}{dt} = 3x_1x_2 - x_1 + x_2, \frac{dx_2}{dt} = x_2^2 - x_1.$$

Solution: The Jacobi matrix has the form:

$$J = \begin{pmatrix} 3x_2 - 1 & 3x_1 + 1 \\ -1 & 2x_2 \end{pmatrix},$$

so

$$J(0,0) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The characteristic equation has the form:

$$(-1-\lambda)(-\lambda) + 1 = \lambda^2 + \lambda + 1 = 0,$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

This is a stable spiral, so the equilibrium point is stable.

6. (25 points)

$$\frac{dx_1}{dt} = -2\sin x_1, \frac{dx_2}{dt} = -x_2 e^{x_1}.$$

Solution: The Jacobi matrix has the form:

$$J = \begin{pmatrix} -2\cos x_1 & 0 \\ -x_2 e^{x_1} & -e^{x_1} \end{pmatrix},$$

so

$$J(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues are equal to -2 and -1, so this is a sink and the equilibrium pojnt is stable.

11. (25 points) Find all equilibria and analyze their stability:

$$\frac{dx_1}{dt} = x_1 - x_2, \frac{dx_2}{dt} = x_1 x_2 - x_2.$$

Solution: The equilibrium points are defined by the equations:

$$x_1 - x_2 = 0,\ x_1 x_2 - x_2 = x_2(x_1 - 1) = 0.$$

From the first equation, $x_1 = x_2$, and from the second $x_1 = 1$ or $x_2 = 0$. Therefore, there are two equilibrium points $(0, 0)$ and $(1, 1)$.

The Jacobi matrix has the form:

$$J = \begin{pmatrix} 1 & -1 \\ x_2 & x_1 - 1 \end{pmatrix},$$

so

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

This is a triangular matrix with eigenvalues 1 and -1, so the equilibrium is a saddle point and hence unstable. Furthermore,

$$J(1,1) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation has the form:

$$(1 - \lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1 = 0,$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}.$$

This is an unstable spiral, so the equilibrium point is unstable.