MAT 17C, Fall 2017
Solutions to homework 7

Chapter 11.3: 14. (25 points) Assume that $a > 0$. Find all point equilibria of

$$\frac{dx_1}{dt} = 1 - ax_1x_2, \quad \frac{dx_2}{dt} = ax_1x_2 - x_2$$

and characterize their stability.

Solution: To find the equilibrium points, we write

$$1 - ax_1x_2 = ax_1x_2 - x_2 = 0,$$

so

$$1 = ax_1x_2 = x_2.$$

Therefore $x_2 = 1$ and from the first equation $x_1 = 1/a$, so there is only one equilibrium point $(1/a, 1)$.

The general Jacobian matrix of the system has the form:

$$J = \begin{pmatrix} -ax_2 & -ax_1 \\ ax_2 & ax_1 - 1 \end{pmatrix},$$

so

$$J(1/a, 1) = \begin{pmatrix} -a & -1 \\ a & 0 \end{pmatrix}.$$

The characteristic equation has the form

$$(-a - \lambda)(-\lambda) + a = \lambda^2 + a\lambda + a = 0.$$

By using the quadratic formula, we get

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4a}}{2}.$$

If $a > 4$ then $a^2 - 4a > 0$, and both eigenvalues are real and negative, so this equilibrium point is a sink. If $0 < a < 4$ then $a^2 - 4a < 0$, and both eigenvalues are complex with negative real part, so this equilibrium point is a stable spiral. In both cases, the equilibrium is stable.

Chapter 11.4: 2. (25 points) Suppose the densities of two species evolve in accordance with the LotkaVolterra model of interspecific competition. Assume that species 1 has intrinsic rate of growth $r_1 = 4$ and carrying capacity $K_1 = 17$ and that species 2 has intrinsic rate of growth $r_2 = 1.5$ and carrying capacity $K_2 = 32$. Furthermore, assume that 15 individuals of species 2 have the same effect on species 1 as 7 individuals of species 1 have on themselves and that 5 individuals of species 1 have the same effect on species 2 as 7 individuals of species 2 have on themselves. Find a system of differential equations that describes this situation.

Solution:

$$\frac{dN_1}{dt} = 4N_1(1 - \frac{N_1}{17} - \frac{7}{15} \cdot \frac{N_2}{17}),$$
$$\frac{dN_2}{dt} = 1.5N_2(1 - \frac{N_2}{32} - \frac{7}{5} \cdot \frac{N_1}{17}).$$

10. (25 points) Find and analyze all equilibria of the following system:

$$\frac{dN_1}{dt} = N_1(1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{25}),$$
$$\frac{dN_2}{dt} = N_2(1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28}).$$

Solution: To find all equilibrium points, we need to set the right hand sides of the equations equal to 0. There are 4 possible cases:

a) $N_1 = N_2 = 0$, the equilibrium point is $(0, 0)$.

b) $N_1 = 0, 1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 0$. It is easy to see from the second equation that in this case $N_2 = 28$, so the equilibrium point is $(0, 28)$.

c) $1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{25} = 0, N_2 = 0$. It is easy to see from the first equation that in this case $N_1 = 25$, so the equilibrium point is $(25, 0)$.

d) $1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{25} = 0, 1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 0$. From the first equation, we can write

$$\frac{N_1}{25} = 1 - 0.1\frac{N_2}{25} \Rightarrow N_1 = 25 - 0.1N_2.$$

We can plug in this into the second equation:

$$1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 1 - \frac{N_2}{28} - \frac{1.2}{28}(25 - 0.1N_2) = 0,$$
$$1 - \frac{1.2 \cdot 25}{28} = N_2(\frac{1}{28} - \frac{1.2 \cdot 0.1}{28}),$$
$$1 - \frac{30}{28} = N_2 \cdot \frac{1 - 0.12}{28}, \ -\frac{2}{28} = N_2 \cdot \frac{0.88}{28},$$

so $N_2 = -2/0.88 < 0$. Since the negative population is impossible, we do not consider this equilibrium.

Next, we need to compute the Jacobi matrix:

$$J = \begin{pmatrix} 1 - 2\frac{N_1}{25} - 0.1\frac{N_2}{25} & -0.1\frac{N_1}{25} \\ -1.2\frac{N_2}{28} & 1 - 2\frac{N_2}{28} - 1.2\frac{N_1}{28} \end{pmatrix}.$$

If we evaluate it at the equilibrium points, we get the following:

a)

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

this is an unstable node (source).

b)

$$J(0, 28) = \begin{pmatrix} 1 - 0.1\frac{28}{25} & 0 \\ -1.2 & 1 - 2\frac{28}{28} \end{pmatrix} = \begin{pmatrix} 1 - 0.1\frac{28}{25} & 0 \\ -1.2 & -1 \end{pmatrix},$$

This is a triangular matrix with eigenvalues $1 - 0.1\frac{28}{25} > 0$ and $-1 < 0$, so this is a saddle point.

c)

$$J(25,0) = \begin{pmatrix} 1 - 2\frac{25}{25} & -0.1 \\ 0 & 1 - 1.2\frac{25}{28} \end{pmatrix} = \begin{pmatrix} -1 & -0.1 \\ 0 & -\frac{2}{28} \end{pmatrix}.$$

This is a triangular matrix with eigenvalues $-\frac{2}{28} > 0$ and $-1 < 0$, so this is stable node (sink).

16. (25 points) Find the equilibrium points and study their stability for the system

$$\frac{dN}{dt} = 5N - PN, \ \frac{dP}{dt} = PN - P.$$

Solution: To find all equilibrium points, we need to set the right hand sides of the equations equal to 0. We have:

$$5N - PN = N(5-P) = 0, \ PN - P = P(N-1) = 0.$$

If $N = 0$ then $P = 0$, and if $P = 5$ then $N = 1$. Therefore the system has two equilibrium points $(0,0)$ and $(1,5)$.

The Jacobi matrix has the form:

$$J = \begin{pmatrix} 5 - P & -N \\ P & N - 1 \end{pmatrix}.$$

Now

$$J(0,0) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix},$$

so $(0,0)$ is a saddle point. At the point $(1,5)$ we have:

$$J(1,5) = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix},$$

the characteristic equation has the form:

$$\lambda^2 + 5 = 0, \lambda = \pm\sqrt{5}i.$$

The eigenvalues are complex with no real part, so this is a center.