Chapter 11.3: 14. (25 points) Assume that $a > 0$. Find all point equilibria of

$$\frac{dx_1}{dt} = 1 - ax_1 x_2, \quad \frac{dx_2}{dt} = ax_1 x_2 - x_2,$$

and characterize their stability.

**Solution:** To find the equilibrium points, we write

$$1 - ax_1 x_2 = ax_1 x_2 - x_2 = 0,$$

so

$$1 = ax_1 x_2 = x_2.$$

Therefore $x_2 = 1$ and from the first equation $x_1 = 1/a$, so there is only one equilibrium point $(1/a, 1)$.

The general Jacobian matrix of the system has the form:

$$J = \begin{pmatrix} -ax_2 & -ax_1 \\ ax_2 & ax_1 - 1 \end{pmatrix},$$

so

$$J(1/a, 1) = \begin{pmatrix} -a & -1 \\ a & 0 \end{pmatrix}.$$

The characteristic equation has the form

$$(-a - \lambda)(-\lambda) + a = \lambda^2 + a\lambda + a = 0.$$

By using the quadratic formula, we get

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4a}}{2}.$$

If $a > 4$ then $a^2 - 4a > 0$, and both eigenvalues are real and negative, so this equilibrium point is a sink. If $0 < a < 4$ then $a^2 - 4a < 0$, and both eigenvalues are complex with negative real part, so this equilibrium point is a stable spiral. In both cases, the equilibrium is stable.

Chapter 11.4: 2. (25 points) Suppose the densities of two species evolve in accordance with the LotkaVolterra model of interspecific competition. Assume that species 1 has intrinsic rate of growth $r_1 = 4$ and carrying capacity $K_1 = 17$ and that species 2 has intrinsic rate of growth $r_2 = 1.5$ and carrying capacity $K_2 = 32$. Furthermore, assume that 7 individuals of species 2 have the same effect on species 1 as 7 individuals of species 1 have on themselves and that 5 individuals of species 1 have the same effect on species 2 as 7 individuals of species 2 have on themselves. Find a system of differential equations that describes this situation.
Solution:
\[
\frac{dN_1}{dt} = 4N_1 \left(1 - \frac{N_1}{17} - \frac{7}{15} \cdot \frac{N_2}{17}\right),
\]
\[
\frac{dN_2}{dt} = 1.5N_2 \left(1 - \frac{N_2}{32} - \frac{7}{5} \cdot \frac{N_1}{17}\right).
\]

10. (25 points) Find and analyze all equilibria of the following system:
\[
\frac{dN_1}{dt} = N_1 \left(1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{25}\right),
\]
\[
\frac{dN_2}{dt} = N_2 \left(1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28}\right).
\]

Solution: To find all equilibrium points, we need to set the right hand sides of the equations equal to 0. There are 4 possible cases:

a) \(N_1 = N_2 = 0\), the equilibrium point is \((0, 0)\).

b) \(N_1 = 0, 1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 0\). It is easy to see from the second equation that in this case \(N_2 = 28\), so the equilibrium point is \((0, 28)\).

c) \(1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{28} = 0, N_1 = 0\). It is easy to see from the first equation that in this case \(N_1 = 25\), so the equilibrium point is \((25, 0)\).

d) \(1 - \frac{N_1}{25} - 0.1 \cdot \frac{N_2}{28} = 0, 1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 0\). From the first equation, we can write
\[
\frac{N_1}{25} = 1 - 0.1 \cdot \frac{N_2}{25} \Rightarrow N_1 = 25 - 0.1N_2.
\]

We can plug this into the second equation:
\[
1 - \frac{N_2}{28} - 1.2 \cdot \frac{N_1}{28} = 1 - \frac{N_2}{28} - \frac{1.2}{28} (25 - 0.1N_2) = 0,
\]
\[
1 - \frac{1.2 \cdot 25}{28} = N_2 \left(1 - \frac{1.2}{28} - \frac{0.1}{28}\right),
\]
\[
1 - \frac{30}{28} = N_2 \cdot \frac{1 - 0.12}{28}, \quad \frac{2}{28} = N_2 \cdot \frac{0.88}{28},
\]
so \(N_2 = -2/0.88 < 0\). Since the negative population is impossible, we do not consider this equilibrium.

Next, we need to compute the Jacobi matrix:
\[
J = \begin{pmatrix} 1 - 2 \frac{N_1}{25} - 0.1 \frac{N_2}{25} & -0.1 \frac{N_1}{25} \\ -1.2 \frac{N_2}{28} & 1 - 2 \frac{N_2}{28} - 1.2 \frac{N_1}{28} \end{pmatrix}.
\]

If we evaluate it at the equilibrium points, we get the following:

a) \(J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\),
this is an unstable node (source).

b) \(J(0, 28) = \begin{pmatrix} 1 - 0.1 \frac{28}{25} & 0 \\ -1.2 & 1 - 2 \frac{28}{28} \end{pmatrix} = \begin{pmatrix} 1 - 0.1 \frac{28}{25} & 0 \\ -1.2 & -1 \end{pmatrix}\),
This is a triangular matrix with eigenvalues $1 - 0.1 \frac{28}{25} > 0$ and $-1 < 0$, so this is a saddle point.

c) \[
J(25, 0) = \begin{pmatrix} 1 - 2 \frac{25}{25} & -0.1 \\ 0 & 1 - 1.2 \frac{25}{28} \end{pmatrix} = \begin{pmatrix} -1 & -0.1 \\ 0 & -\frac{2}{28} \end{pmatrix}.
\]
This is a triangular matrix with eigenvalues $-\frac{2}{28} > 0$ and $-1 < 0$, so this is stable node (sink).

16. (25 points) Find the equilibrium points and study their stability for the system
\[
\frac{dN}{dt} = 5N - PN, \quad \frac{dP}{dt} = PN - P.
\]

**Solution:** To find all equilibrium points, we need to set the right hand sides of the equations equal to 0. We have:

\[
5N - PN = N(5 - P) = 0, \quad PN - P = P(N - 1) = 0.
\]

If $N = 0$ then $P = 0$, and if $P = 5$ then $N = 1$. Therefore the system has two equilibrium points $(0, 0)$ and $(1, 5)$.

The Jacobi matrix has the form:
\[
J = \begin{pmatrix} 5 - P & -N \\ P & N - 1 \end{pmatrix}.
\]

Now
\[
J(0, 0) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix},
\]
so $(0, 0)$ is a saddle point. At the point $(1, 5)$ we have:
\[
J(1, 5) = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix},
\]
the characteristic equation has the form:
\[
\lambda^2 + 5 = 0, \quad \lambda = \pm \sqrt{5}i.
\]
The eigenvalues are complex with no real part, so this is a center.