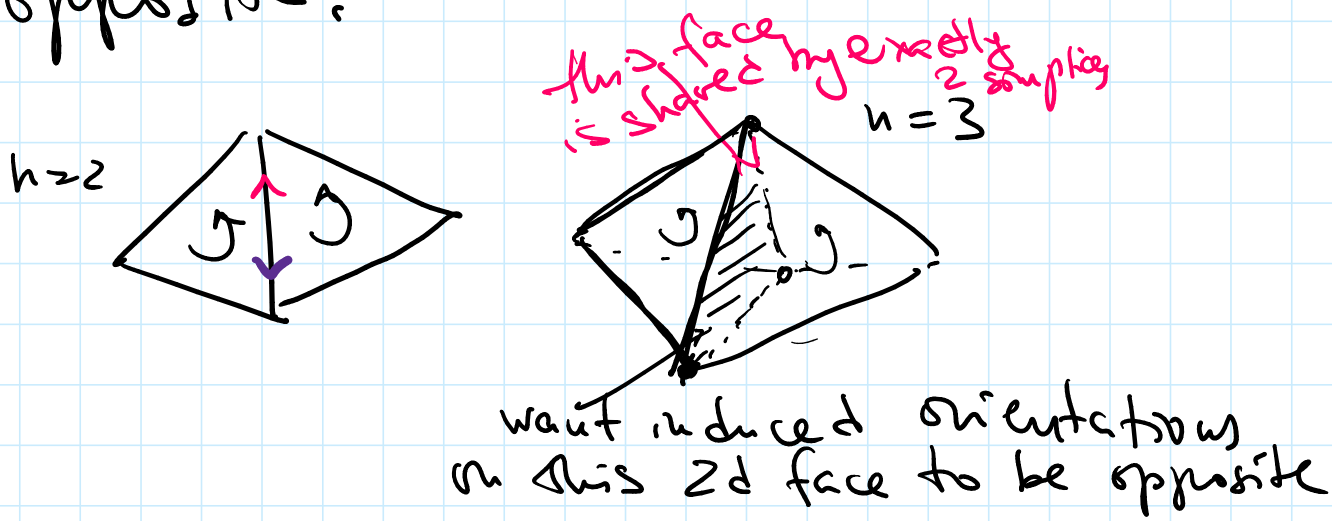
Lecture 11 (4/21)

Wednesday, April 21, 2021 2:03 PM

Orientations: general case

(1) PL manifolds M = n-dim PL manifold

M is orientable if can choose orientation for each n-simplex such that induced orientations on each (n-1)-simplex are opposite.



Fact If M is a PL manifold then each (n-1)-dim simplex bounds exactly 2 n-simplices.

Similarly to surfaces:

Similarly to surfaces:

1. If M is orientable and connected, then M has exactly 2 orientations.
2. <u>Thm</u> Let M be a closed (compact, <u>connected</u>) PL manifold. Then

   (a) If M is orientable, then $H_n(M,\mathbb{Z})=\mathbb{Z}$

   (b) If M is non-orientable, then $H_n(M,\mathbb{Z})=0$

   (c) For <u>any</u> M, $H_n(M,\mathbb{Z}_2)=\mathbb{Z}_2$

<u>Cor</u>: By universal coefficient thm,

(a) $H^n(M,\mathbb{Z})=\mathbb{Z}$ (b) $H^n(M,\mathbb{Z})=\mathbb{Z}_2$ (c) $H^n(M,\mathbb{Z}_2)=\mathbb{Z}_2$.

<u>Proof</u>: same as for surfaces

M oriented $\to$ <u>fundamental class</u>

$[M]\in H_n(M,\mathbb{Z})$

$[M]=$ sum of all n-simplices with chosen orientation

$\partial[M]=0$ since every $(n-1)$-face will be counted twice in $\partial[M]$

with opposite signs by def. of orientation.

$H_n(M, \mathbb{Z})$ = free abelian group generated by $[M]$.

Note: This depends on the choice of orientation on M, opposite orientation gives $-[M]$.

Rmk For any M, we can define fundamental class $[M] \in H_n(M, \mathbb{Z}_2)$ = sum of all n-simplices with coef. $1 \in \mathbb{Z}_2$.

Proof $\partial[M]$ = sum of all $(n-1)$ simplices

each appears twice, either with opposite coefs $\Rightarrow 0$

or with same coef $\Rightarrow \pm 2 = 0$ in $\mathbb{Z}_2$.

$\Rightarrow \partial[M] = 0$ with $\mathbb{Z}_2$ coefs. ∎

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② Smooth manifolds

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth function (C^∞)

$f(x_1 \ldots x_n) = (f_1(x_1 \ldots x_n), \ldots f_n(x_1 \ldots x_n))$

Jacobian: $J_f = \det\left(\frac{\partial f_i}{\partial x_j}\right) \leftarrow$ function.

Jacobian : $J_f = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ ← function on $\mathbb{R}^n$

Fact: If is invertible, $f \circ g = id$, g is smooth then $J_f \neq 0$.

Proof: $J_{f\circ g} = J_f \cdot J_g = 1$ ← chain rule

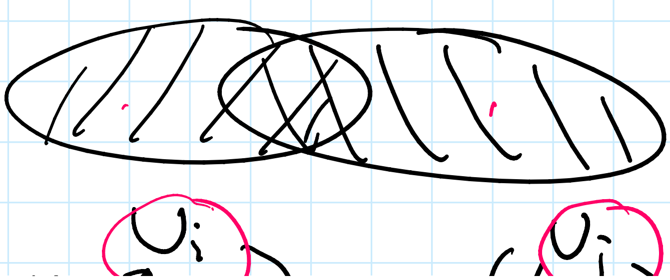
$\Rightarrow J_g = (J_f)^{-1} \Rightarrow J_f \neq 0$.

Def $f: \mathbb{R}^n \to \mathbb{R}^n$ preserves orientation if $J_f > 0$, and reverses orientation if $J_f < 0$.

Rmk If $f(x_1 \ldots x_n) = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ for some matrix A, then $J_f = \det A$.

M = smooth n-manifold

We have a cover by local coordinate charts $U_i \simeq \mathbb{R}^n$



$$
\begin{tikzcd}
 & U_i \arrow[dl, "\varphi_i"'] & U_i \cap U_j \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] \arrow[dl] \arrow[dr] & U_j \arrow[dr, "\varphi_j"] & \\
\mathbb{R}^n & \varphi_i(U_i \cap U_j) \arrow[rr, "\varphi_{ij}"] & & \varphi_j(U_i \cap U_j) & \mathbb{R}^n
\end{tikzcd}
$$

transition function.

local coordinates in U_i

change of coords on $U_i \cap U_j$

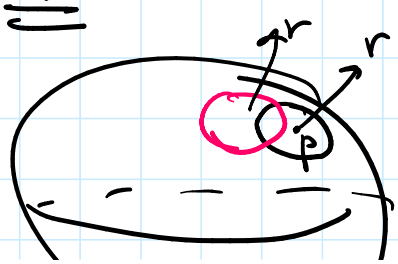
local coords in U_j

M is oriented if φ_{ij} preserve orientation (that is, $J_{\varphi_{ij}} > 0$) for all i and j.

Rmk 1) This is much easier in practice than PL

2) Warning: Reordering local coords can change orientation on U_i (by the sign of permutation).

Ex $S^n \subset \mathbb{R}^{n+1}$



$\begin{pmatrix}\text{local}\\ \text{coords on } S^n\\ \text{at } p\end{pmatrix} \cup r = \begin{pmatrix}\text{local}\\ \text{coord on } \mathbb{R}^{n+1}\\ \text{at } p\end{pmatrix}$

radial

at p radial coord at p)

We can choose local coordinates such that transition functions do not change orientations of $\mathbb{R}^{n+1}$

$$\det \begin{array}{cc} & \begin{array}{cc} S^n & r \end{array} \\ \begin{array}{c} \\ r \end{array} & \left(\begin{array}{c|c} J_{\varphi_{ij}} & 0 \\ \hline 0 & 1 \end{array}\right) \end{array} = \det\left(J_{\varphi_{ij}}\right)$$

If preserves orientation of $\mathbb{R}^{n+1}$

then φ_{ij} preserves orientation of S_n.

<u>Ex</u> $M = \{F = 0\} \subset \mathbb{R}^n$

Assuming M is smooth (and we can apply Implicit Function Thm)

we can write

$$\begin{pmatrix}\text{local coord} \\ \text{in } \mathbb{R}^n\end{pmatrix} = \begin{pmatrix}\text{local} \\ \text{coord in } M\end{pmatrix} \cup \{\underline{F}\}$$

$\Rightarrow M$ is orientable.

<u>Ex</u> $\mathbb{RP}^n = S^n / \{\pm 1\}$

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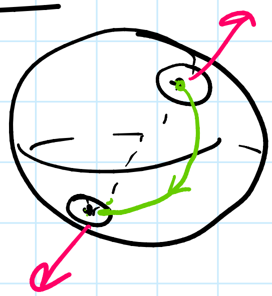


$$(-1) = \underbrace{\begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}}_{n+1} \qquad \det = (-1)^{n+1}$$

This changes orientation of $\mathbb{R}^{n+1}$ by $(-1)^{n+1}$

sends r to r $\Rightarrow$ changes orientation of S^n by $(-1)^{n+1}$

$\Rightarrow$ $\mathbb{RP}^n$ is orientable if $n+1$ is even $\Leftrightarrow$ n odd   ex. $S^1 \cong \mathbb{RP}^1$

$\mathbb{RP}^n$ is not orientable if $n+1$ is odd $\Leftrightarrow$ n even

$\boxed{\text{ex: } \mathbb{RP}^2}$

③ Topological manifolds

What is an orientation is no simplices/Jacobian?

$$H_n(M; M - \{x\}) \underset{\text{excision}}{=} H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) =$$

$$= \tilde{H}_{n-1}(\mathbb{R}^n - \{x\}) = \tilde{H}_{n-1}(S^{n-1}) \simeq \mathbb{Z}$$

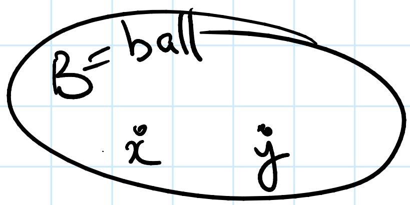
exact sequence of a pair

"local cohomology"

sequence pair

Local orientation at x: generator of this $\mathbb{Z} = H_n(M; M - \{x\})$

Two possible choices of a generator $\Rightarrow$ two local orientations.



Def Local orientations at x and y agree if

$$\begin{tikzcd} & H_n(M; M-B) \arrow[dl, "\wr"] \arrow[dr, "\wr"] & \\ H_n(M; M-\{x\}) & & H_n(M; M-\{y\}) \end{tikzcd}$$

Orientations at x and y agree if there's a class in $H_n(M; M-B)$ which projects to both.

Note: Does not depend on a choice of B.

Rmk Depends on coefficients!

R-orientation for any coefficients ring R, just use $H_n(-; R)$

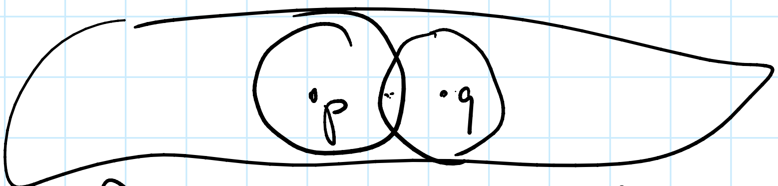
$H_n(M; M-\{x\}; R) \simeq R$ as above. free R-module of rank 1.

Def M is (R-)orientable if we

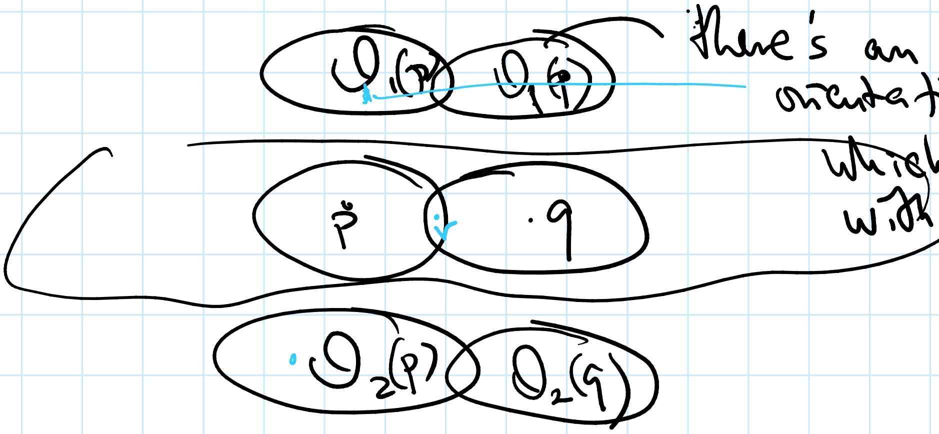
of rank 1.

<u>Def</u> M is (R-)orientable if we can choose local orientation at every point such that for any pair of points in a ball B, orientations agree.

Orientation cover $\tilde{M} \longrightarrow M$



$\tilde{M} = \{p \in M, \text{ local orientation at } p\}$



there's an local orientation at q which agrees with orientation at p

HW3: If M is not orientable connected
prove that $\tilde{M}$ is connected.