Lecture 20 (5/12)

Wednesday, May 12, 2021 2:03 PM

Alexander duality

Thm M = ~~closed~~ compact, orientable n-manifold

$K \subset M$ submanifold (~~compact~~ OK, locally contractible)

Then

$$\boxed{H_i(M; M-K) \simeq H^{n-i}(K)}$$

Proof: Hatcher:

Ex: $K = \{pt\}$

$$H_i(M; M-pt) \simeq H^{n-i}(pt) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$$

$\|$ by excision

$H_i(\mathbb{R}^n, \mathbb{R}^n - pt)$

$\Leftrightarrow$ homol. definition of local orientation.

Thm (Alexander duality)

$K \subset S^n$ as before

Then $\boxed{\tilde{H}_i(S^n - K) \simeq \tilde{H}^{n-i-1}(K)}$

Rmk: $S^n - K$ can have really complicated topology (ex. complicated π_1).

Proof: Long exact sequence:

$$\to H_{i+1}(S^n) \to H_{i+1}(S^n; S^n - K) \to H_i(S^n - K) \to$$

$$\to H_i(S^n) \to H_i(S^n; S^n - K) \to H_{i-1}(S^n - K) \to \dots$$

If $i \neq n, n-1, 0$ then $H_i(S^n) = 0$

$H_{i+1}(S^n) = 0$

$$H_i(S^n - K) \cong H_{i+1}(S^n; S^n - K) = H^{n-i-1}(K)$$

by previous thm.

Special cases: $i = 0$

$$0 \to H_1(S^n; S^n - K) \to H_0(S^n - K) \to \overset{\mathbb{Z}}{\overset{\parallel}{H_0(S^n)}}$$

$$\to \underset{0}{\underset{\parallel}{H_0(S^n; S^n - K)}} \to 0$$

reduced $\tilde{H}_0(S^n - K) \simeq H_1(S^n; S^n - K) \simeq H^{n-1}(K)$

$(i = n-1)$

$$0 \to \underset{\underset{\mathbb{Z}}{\parallel}}{H_n(S^n)} \to H_n(S^n; S^n - K) \to H_{n-1}(S^n - K) \to \underset{\underset{0}{\parallel}}{H_{n-1}(S^n)}$$

$$H_{n-1}(S^n - K) \oplus \mathbb{Z} \simeq \underset{\underset{H^0(K)}{\cong}}{H_n(S^n; S^n - K)}$$

$$H_{n-1}(S^n - K) \simeq \tilde{H}^0(K)$$

kill extra $\mathbb{Z}$ in reduced homology.

Rmk We can use $\mathbb{R}^n$ instead of S^n

$$\mathbb{R}^n = S^n - pt \qquad \text{if } K \subset \mathbb{R}^n$$

$$\mathbb{R}^n - K = \mathbb{R}^n - (K \sqcup pt).$$

Ex $\mathbb{R}^n - pt = S^n - 2\,pts \sim S^{n-1}$

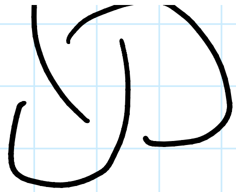
$$H_{n-1}(\mathbb{R}^n - pt) = H_{n-1}(S^n - 2\,pts)$$

$\parallel$ Alexander

$$\tilde{H}^0(2\,pts) = \mathbb{Z}.$$

Ex K = knot in S^3 or $\mathbb{R}^3$

$$S^1 \hookrightarrow S^3$$



smooth 1-submanifold of S^3
connected

$S^3 - K$ has really complicated topology!

<u>Rmk</u> $\pi_1(S^3 - K)$ is almost a complete knot invariant.

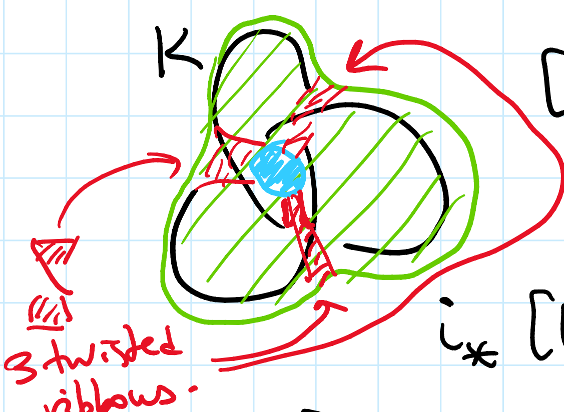
<u>Alexander duality:</u>

$$\tilde{H}_i(S^3 - K) \simeq \tilde{H}^{3-i-1}(K) = \underbrace{\tilde{H}^{2-i}(K)}_{\substack{\text{easy!} \\ (K \simeq S^1)}}$$

$$H_2(S^3 - K) \simeq \tilde{H}^0(K) = 0$$

$$\boxed{H_1(S^3 - K) \simeq H^1(K) = H^1(S^1) = \mathbb{Z}}$$

How to see this more explicitly



$$\begin{array}{c} [K] \in H_1(K) \simeq \mathbb{Z} \\ \downarrow i_* \\ H_1(S^3) \simeq 0 \end{array}$$

$i_*[K] = 0 \Rightarrow$ there

is $J \in C_2(S^3)$ s.t. that

ribbons.

is $\Sigma \in C_2(S^3)$ such that

$\partial \Sigma = i_*[K]$

Fact Σ can be realized by a smooth oriented surface with boundary on K (Seifert surface)

L = class in $H_1(S^3 - K)$

realized by another knot/link in $S^3 - K$.

Pairing:

$lk : H_1(S^3 - K) \times H_1(K) \longrightarrow \mathbb{Z}$

$lk(L, [K]) = (L^1 \cdot \Sigma^2)$ intersection number

linking number

Alexander duality: well defined, does not depend on the choice of Σ

given a nondegenerate pairing.

$lk : H_1(S^3 - K) \times H_1(K) \to \mathbb{Z}$

Ex

Hopf link

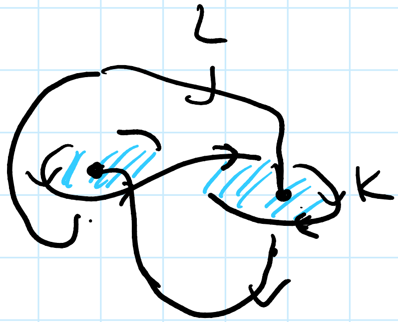
Ex

Hopf link.

$$L \cap \Sigma = 1 \text{ point}$$

$L \cdot \Sigma = \pm 1$ depending on orientations.

Ex



Whitehead link

2 intersection points with opposite signs.

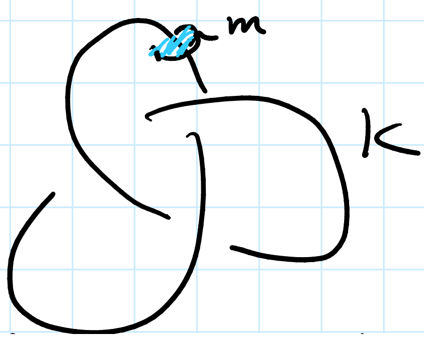
$$lk(K, L) = \Sigma \cdot L = 0$$

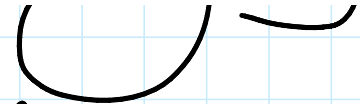
Fact: $lk(K, L) = lk(L, K)$

$\partial\Sigma = K$, $\Sigma \cdot L$

$\partial\Sigma' = L$, $\Sigma' \cdot K$

$H_1(S^3 - K) \cong \mathbb{Z}$, what is the generator? The meridian for K





m = small loop around one strand of K

(Disk bounding m) $\cdot K$ = 1 pt.

$\Rightarrow lk(m, K) = 1$ (with correct orientations)

$\Rightarrow m$ generates $H_1(S^3 - K)$

and m intersects any Seifert surface of K at one point.

This generalizes to higher dimensions $\rightsquigarrow$ higher dimensional knots & linking numbers.

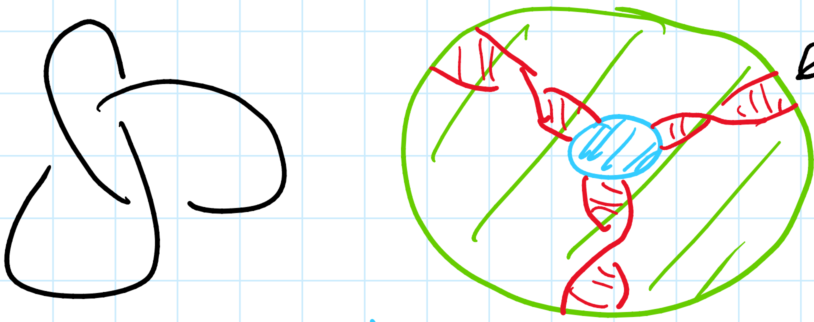
$S^2 \hookrightarrow S^5$ can be linked

<u>Alexander</u>: $H_2(S^5 - \text{knotted } S^2) =$

$= H^{5-2-1}(S^2) = H^2(S^2) = \mathbb{Z}$

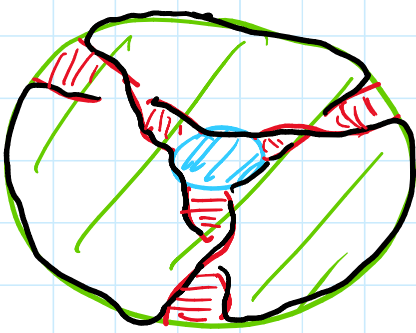
$lk: H_2(S^5 - \text{knotted } S^2) \times H_2(S^2) \to \mathbb{Z}$.

Seifert surface for trefoil



surface Σ with boundary

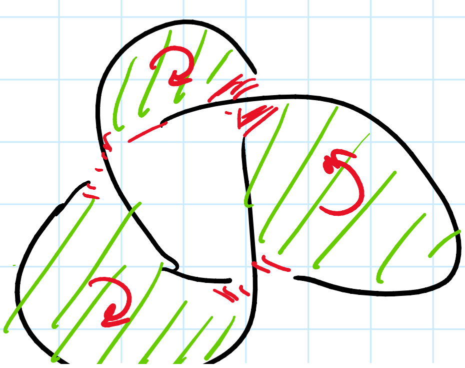
blue disk on top of green disk.



boundary of Σ

<u>Def</u> A Seifert surface for K = connected, oriented smooth surface $\Sigma \subset S^3$, such that $\partial\Sigma = K$.

<u>Thm</u> (Seifert) Such Σ exists.



not orientable!

change orientation for each ribbon,

for each ribbon,
contradiction.