Lecture 25 (5/24)

Monday, May 24, 2021 2:09 PM

Vector bundle: $E \xrightarrow{\pi} M$

- $\pi^{-1}(p)$ is a vector space of dimension n
- Locally trivial: for any $p \in M$ there is a nbhd U such that

$$\pi^{-1}(U) = \underset{(p,\,v)}{U \times \mathbb{R}^n}$$

respecting the vector space structure

$$(p, v) + (p, v') = (p, v + v')$$

and so on

can add vectors only if they project to the same point.

<u>Ex</u> TS^n tangent bundle of S^n

$$S^n = \{(x_1 \dots x_{n+1}) : \underline{\sum x_i^2 = 1}\} \subset \mathbb{R}^{n+1}$$

$TS^n = \{(p, v) : p \in S^n$

v is a tangent vector to S^n at p

$\Updownarrow$

$p = (x_1, \dots x_{n+1})$

$v = (v_1, \dots, v_{n+1})$

$r = (x_1, \dots, x_{n+1})$

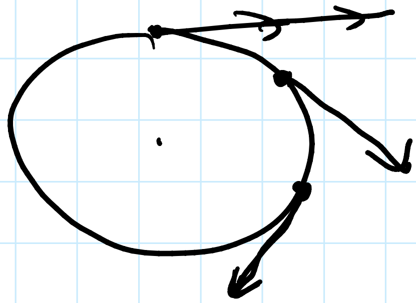
$v = (v_1, \dots, v_{n+1})$

v is a tangent vector $\iff$ it is perpendicular to the radius

$\iff$

$$x_1 v_1 + \dots + x_{n+1} v_{n+1} = 0.$$

TS^1



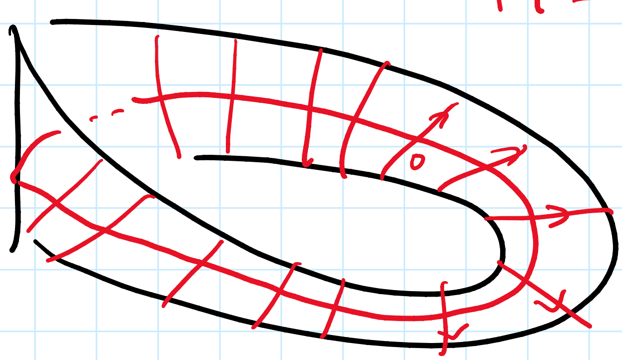
Rank $= n$

$$\dim E = \underbrace{n}_{\dim(\text{fiber})} + \dim S^n = 2n$$

Ex Mobius band

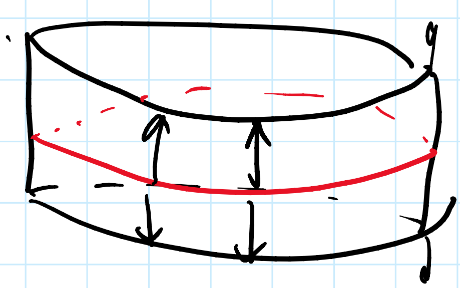
$M \longrightarrow S^1$

$=$ middle circle



do not include the boundary

fibers $\simeq \mathbb{R}$



Cylinder $\to S^1$

fibers $\simeq \mathbb{R}$

Cylinder $\simeq S^1 \times \mathbb{R}$ trivial

Cylinder $\simeq S^1 \times \mathbb{R}$ trivial vector bundle

Mobius band is nontrivial.

<u>Fact</u> (a) TS^1 is trivial,

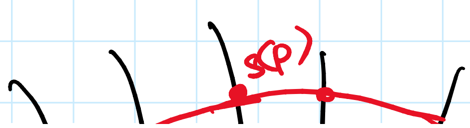
$$TS^1 \simeq \text{cylinder} \quad \text{(HW)}$$

(b) In fact, any vector bundle over S^1 with fiber $\mathbb{R}$ is either isomorphic to a Mobius band, or to a cylinder = trivial.

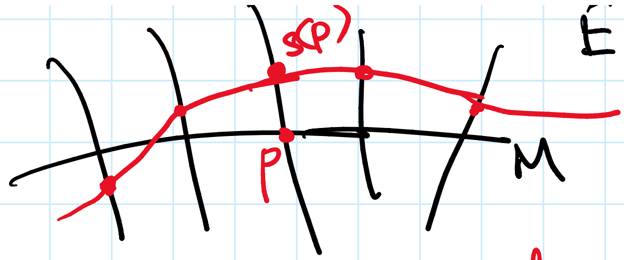
How to check that a vector bundle is trivial

trivial $\simeq M \times \mathbb{R}^n$

<u>Def</u> A section of a vector bundle $\pi: E \longrightarrow M$ is a function $s: M \to E$ such that $\pi(s(p)) = p$ for all p.



E $\Leftrightarrow$ for every



$\Leftrightarrow$ for every p in M, choose a vector (possibly 0) in $\pi^{-1}(p)$.

Ex $s \equiv 0$ zero section

$s(p) = 0$ for all p is a section

Thm A rank n vector bundle is trivial if and only if it has n sections $s_1, \dots, s_n$ which are linearly independent at every point of M.

Cor A rank 1 bundle is trivial iff it has a section which does not vanish anywhere.

Proof 1) Suppose that E is trivial

$$E = M \times \underline{\mathbb{R}^n}$$

Choose a basis in $\mathbb{R}^n$ $e_1 \dots e_n$

Choose a basis in $\mathbb{R}^n$ $e_1 \dots e_n$

these are the desired section
(all fibers $\simeq \mathbb{R}^n$)

2) $\underset{x}{E} \longrightarrow M$ $\quad s_1 \dots s_n$ indep. sections

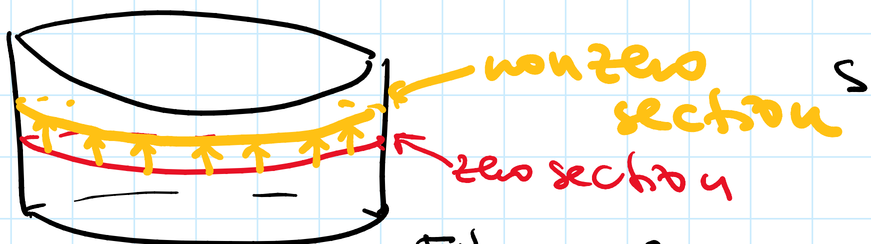
$\pi(x) = p$ $\qquad s_1(p), \dots s_n(p) = $ basis in $\pi^{-1}(p)$

$x = \alpha_1 s_1(p) + \dots + \alpha_n s_n(p)$

$$\begin{tikzcd} E \arrow[rr, dashed, "\varphi"] \arrow[dr] & & M \times \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

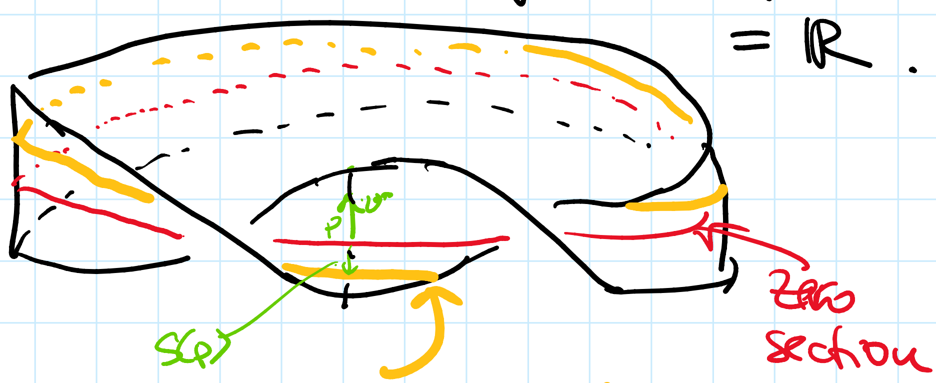
$\varphi(x) = (p; \underbrace{\alpha_1 \dots \alpha_n}_{\mathbb{R}^n})$

Ex



Fiber over any point $= \mathrm{Span}(s)$
$= \mathbb{R}$.

Ex



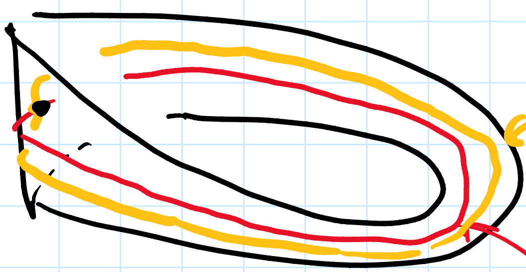
$v = \alpha \cdot s(p)$

because $s(p) \neq 0$ we can do this

$$(\text{2-twisted ribbon}) \Longrightarrow S^1 \times \mathbb{R}$$

$$v \longrightarrow (p, \alpha)$$

Ex



can construct a section which vanishes at one point

zero section

$$(\underbrace{\text{yellow section}}_{s}) \circ (\text{zero section}) = 1 \bmod 2$$

mod 2 intersection form.

Invariance of intersection form mod 2

$\Rightarrow$ any section intersects zero section at odd number of points $\Rightarrow$ has odd number of zeros $\Rightarrow$

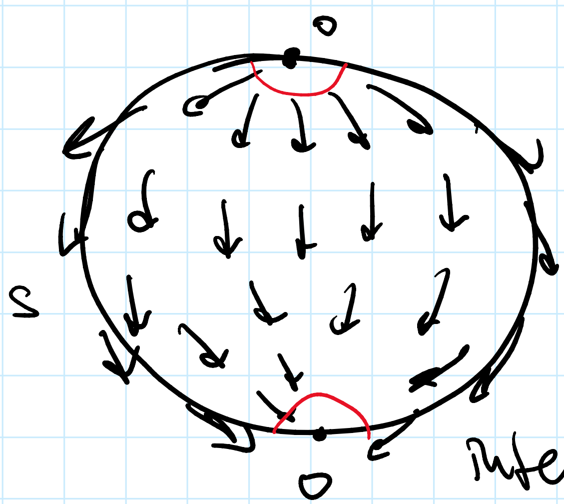
$\Rightarrow$ <u>no</u> nonzero section $\Rightarrow$ this bundle is nontrivial.

Ex TS^2, what is a section of it? $s(p) = (p, v)$ where v is a tangent vector at p

$s: S^2 \longrightarrow TS^2$
section

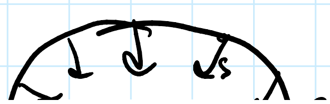
For any point, we need to choose a tangent vector

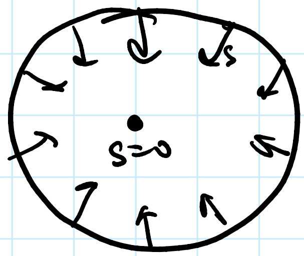
$\Rightarrow$ vector field!



Can use the same idea. compute the intersection index between zero section and a generic section s = number of zeros of s with signs.

Signs (in 2d)





small circle around a zero

$$\underbrace{\frac{s^{\in \mathbb{R}^2}}{\|s\|}}_{} : S^1 \longrightarrow S^1 = \{u : \|u\| = 1\}$$

vector field s

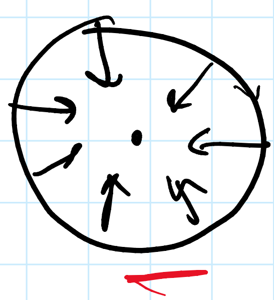
length of s.

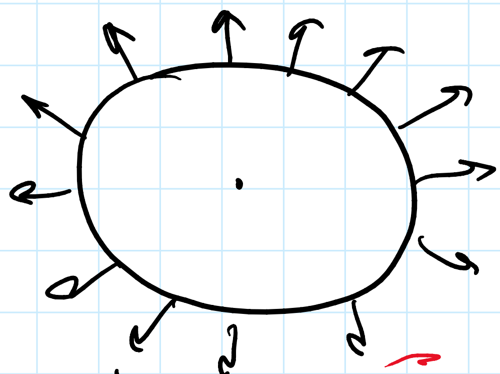
How many times s rotates as we go around.

Index of a zero point of a vector field = degree of this map $\frac{s}{\|s\|} : S^1 \longrightarrow S^1$

Sign = index $(= \pm 1)$

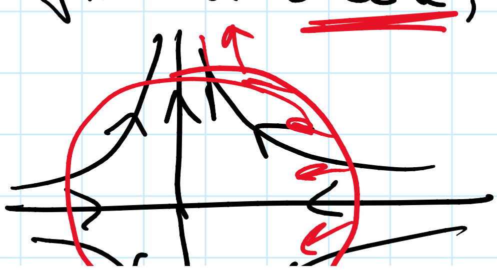
For S^2 and section above.

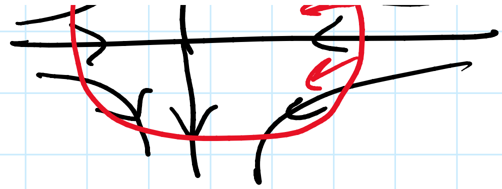




index = 1 = index

(note: for a saddle, index = -1





Conclusion: (generic section) $\circ$ (zero section)

= sum of indices of zero points

of a vector field

This does not depend on the choice of

generic section (mod 2 in general,
over $\mathbb{Z}$ if TM is oriented)

For TS^2 we get

(generic section) $\cdot$ (zero section)
$= 1 + 1 = 2 \neq 0$

$\Rightarrow$ Any section of TS^2 (or any
vector field on S^2) has a zero!

$\Rightarrow$ TS^2 is a nontrivial bundle

$\underset{\text{zero section}}{S^2} \subset TS^2$ $\qquad$ $\underline{[S^2] \cdot [S^2]}$

$H_2(TS^2) \times H_2(TS^2) \to \mathbb{Z}$

$s : S^2 \longrightarrow TS^2$

$s_0 : S^2 \longrightarrow TS^2$

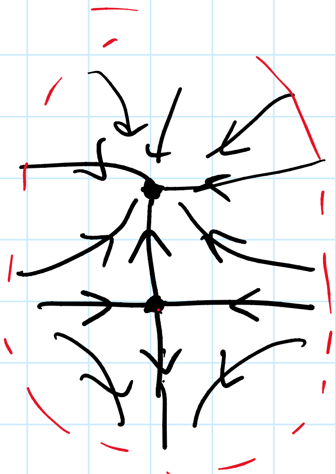
$p \longrightarrow (p, 0)$ zero section

$s : S^2 \longrightarrow TS^2$

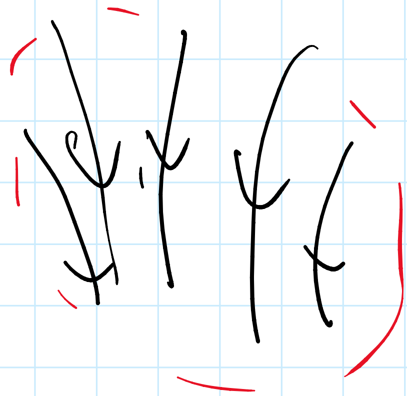
$p \longrightarrow (p, v(p))$

$s_0(S^2) \cap s(S^2) = \{p : v(p) = 0\}$

set of zeros of v



$1 + (-1) = 0$



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