MAT 21B, Fall 2016
Solutions to Homework 3

Section 5.5 Compute the following integrals:
34. (10 points) $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3)dt$.

Solution: Let $u = \sqrt{t}+3$, then $du = \frac{1}{2\sqrt{t}}dt$ and

$$\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3)dt = 2\int \cos(u)du = 2\sin(u) + C = 2\sin(\sqrt{t}+3) + C.$$

44. (10 points) $\int x\sqrt{4-x}dx$

Solution: Let $u = 4-x$, then $x = 4-u$ and $du = -dx$, therefore

$$\int x\sqrt{4-x}dx = -\int (4-u)\sqrt{u}du = -\int (4u^{1/2} - u^{3/2})du = -4\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + C =$$

$$-\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + C.$$

58. (10 points) $\int \frac{dx}{x\sqrt{x^4-1}}$

Solution: Let $u = x^4-1$, then $du = 4x^3dx$ and $\frac{dx}{x} = \frac{du}{4x^4} = \frac{du}{4(u+1)}$. Therefore

$$\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{4}\int \frac{du}{(u+1)\sqrt{u}}.$$

To compute this integral, one needs to make another change of variables. Let $v = \sqrt{u}$, then $dv = \frac{du}{2\sqrt{u}}$ and $u+1 = v^2+1$. Therefore:

$$\frac{1}{4}\int \frac{du}{(u+1)\sqrt{u}} = \frac{1}{4}\int \frac{2dv}{(v^2+1)} = \frac{1}{2}\int \frac{dv}{v^2+1} = \frac{1}{2}\arctan(v) + C.$$

Since $v = \sqrt{u} = \sqrt{x^4-1}$, we conclude that

$$\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\arctan(\sqrt{x^4-1}) + C.$$

Remarks: a) One may instead consider the change of variables $v = \sqrt{x^4 - 1}$ directly, then by Chain Rule $dv = \frac{4x^3dx}{2\sqrt{x^4-1}} = \frac{2x^3dx}{\sqrt{x^4-1}}$ and

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{dv}{2x^3 \cdot x} = \int \frac{dv}{2x^4} = \int \frac{dv}{2(v^2+1)}.$$

b)Using trigonometric identities, one could write the answer differently. Indeed, let $\alpha = \arctan(\sqrt{x^4-1})$, then $\tan(\alpha) = \sqrt{x^4-1}$. Now

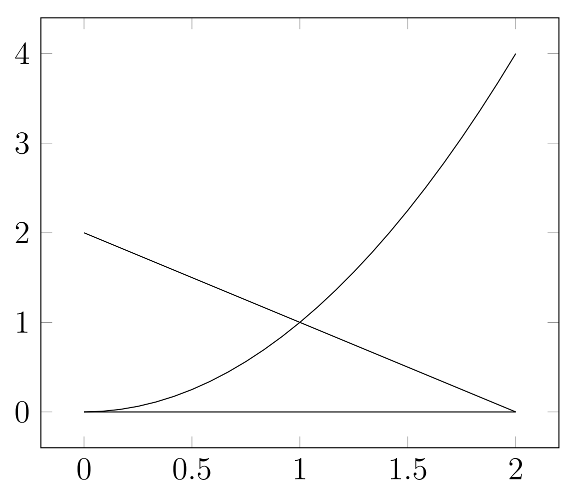
$$x^4 = \tan^2(\alpha) + 1 = \frac{\sin^2(\alpha)}{\cos^2(\alpha)} + 1 = \frac{\sin^2(\alpha) + \cos^2(\alpha)}{\cos^2(\alpha)} = \frac{1}{\cos^2(\alpha)},$$

so $\cos^2(\alpha) = \frac{1}{x^4}$, and we can assume $\cos(\alpha) = \frac{1}{x^2}$. Therefore:

$$\arctan(\sqrt{x^4-1}) = \arccos(\frac{1}{x^2}) = \sec^{-1}(x^2),$$

$$\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\arccos(\frac{1}{x^2}) = \frac{1}{2}\sec^{-1}(x^2).$$

Section 5.6 58. (10 points) Find the area of the shape bounded by the curves $y = 0, y = x^2$ and $x + y = 2$.



Solution: Let us find the intersection points of the parabola $y = x^2$ and the line $x + y = 2$. We get: $x + x^2 = 2$, $x^2 + x - 2 = (x-1)(x+2) = 0$, so the parabola intersects the line twice: at $x = -2$ and at $x = 1$. The first point is outside the domain of integration, and

$$\text{Area} = \int_0^1 x^2 dx + \int_1 2(2-x)dx = \frac{x^3}{3}|_0^1 + (2x - \frac{x^2}{2})|_1^2 = 1/3 - 0 + (4-2) - (2-1/2) = 5/6.$$

Remark: One could instead express x in term of y and integrate over y-axis:

$$\text{Area} = \int_0^1 (2 - y - \sqrt{y})dy = (2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2})|_0^1 = 2 - 1/2 - 2/3 = 5/6.$$