MAT 21B, Fall 2016
Solutions to Homework Assignment 5

Section 6.5: 32. (10 points) Two electrons r meters apart repel each other with a force of $F = 23 \cdot 10^{-29}/r^2$ newtons.

a) Suppose one electron is fixed at $(1, 0)$. How much work does it take to move a second electron along the x-axis from $(-1, 0)$ to $(0, 0)$?

b) Suppose two electrons are fixed at $(-1, 0)$ and at $(1, 0)$. How much work does it take to move a third electron along the x-axis from $(5, 0)$ to $(3, 0)$?

Solution: Let $A = 23 \cdot 10^{-29}$ so that $F = A/r^2$. Then in (a) $F(x) = A/(x-1)^2$ and

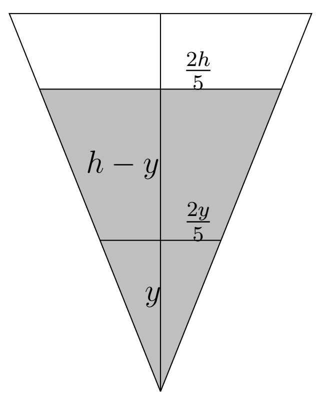
$$W = \int_{-1}^{0} \frac{Adx}{(x-1)^2} = \frac{-A}{x-1}|_{-1}^{0} = A - \frac{A}{2} = \frac{A}{2}.$$

In (b) $F(x) = A/(x-1)^2 + A/(x+1)^2$, so

$$W = \int_{3}^{5} \left(\frac{A}{(x-1)^2} + \frac{A}{(x+1)^2} \right) dx = \left(-\frac{A}{x-1} - \frac{A}{x+1} \right) |_{3}^{5} =$$

$$(-A/4 - A/6) - (-A/2 - A/4) = -A/4 - A/6 + A/2 + A/4 = A/2 - A/6 = A/3.$$

44. (10 points) The end plates of the trough were designed to withstand a force of 6667lb. How many cubic feet of water can the tank hold without exceeding this limitation?



Solution: Suppose that the depth of the water is h. At the level y from the bottom, the width of the triangular plate equals $2 \cdot \frac{2y}{5} = \frac{4y}{5}$ and the depth is $h - y$, so the total fluid force equals:

$$F(h) = \int_0^h w(h-y)\frac{4y}{5}dy = \frac{4w}{5}\int_0^h (hy - y^2)dy = \frac{4w}{5}(hy^2/2 - y^3/3)|_0^h =$$

$$= \frac{4w}{5}(h^3/2 - h^3/3) = \frac{4w}{5} \cdot h^3/6 = \frac{2wh^3}{15}.$$

/ Here $w = \rho \cdot g = 62.4$ lb$/ft^3$ is the weight-density of water. Now

$$F(h) = 6667 \Leftrightarrow \frac{2 \cdot 62.4 \cdot h^3}{15} = 6667 \Leftrightarrow h^3 = \frac{6667 \cdot 15}{2 \cdot 62.4} \approx 801.3,$$

so $h \approx \sqrt[3]{801.3} \approx 9.3$.

Finally, the volume equals

$$V = 30 \cdot \frac{1}{2}h \cdot \frac{4h}{5} = 12h^2 \approx 1037.9.$$

Section 6.6: 11. (10 points) Find the center of mass of the region bounded by the curves $y = \pm\frac{1}{1+x^2}$ and by the lines $x = 0$ and $x = 1$.

Solution: We have

$$M_x = \int_0^1 x(\frac{1}{1+x^2} - \frac{-1}{1+x^2})dx = \int_0^1 \frac{2xdx}{1+x^2}$$

Let $u = 1 + x^2$, then $du = 2xdx$ and

$$M_x = \int_1 2\frac{du}{u} = \ln(u)|_1^2 = \ln 2.$$

Also

$$M_y = \frac{1}{2}\int_0^1 \left(\left(\frac{1}{1+x^2}\right)^2 - \left(\frac{-1}{1+x^2}\right)^2\right) dx = 0$$

and

$$M = \int_0^1 (\frac{1}{1+x^2} - \frac{-1}{1+x^2})dx = \int_0^1 \frac{2dx}{1+x^2} = 2\arctan(x)|_0^1 = 2\arctan(1).$$

Therefore

$$\overline{x} = M_x/M = \frac{\ln(2)}{2\arctan(1)},\ \overline{y} = M_y/M = 0.$$

Remark: Note that $\arctan(1) = \pi/4$.

30. (10 points) Find the centroid of the region bounded by the functions $g(x) = x^2(x+1)$, $f(x) = 2$ and $x = 0$.

Solution: Note that $f(1) = g(1) = 2$. Now

$$M_x = \int_0^1 x(f(x) - g(x))dx = \int_0^1 x(2 - x^2 - x^3)dx = \int_0^1 (2x - x^3 - x^4)dx = (x^2 - x^4/4 - x^5/5)|_0^1 = 11/20,$$

$$M_y = \frac{1}{2}\int_0^1 (f^2(x) - g^2(x))dx = \frac{1}{2}\int_0^1 (4 - x^4 - 2x^5 - x^6)dx = \frac{1}{2}(4x - x^5/5 - 2x^6/6 - x^7/7)|_0^1 = 349/210,$$

$$M = \int_0^1 (f(x) - g(x))dx = \int_0^1 (2 - x^2 - x^3)dx = (2x - x^3/3 - x^4/4)|_0^1 = 17/12.$$

Now

$$\overline{x} = M_x/M = 33/85,\ \overline{y} = M_y/M = 698/595.$$