MAT 21B, Spring 2019
Solutions to homework 4

Section 6.1: 14. (10 points) A solid lies between the planes $x = 0$ and $x = 12$, and its cross-sections are circles whose diameters run from $y = x/2$ to $y = x$. Explain why the solid have the same volume as the circular cone with radius 3 and height 12.

Solution: The diameter of the cross-section equals $x - x/2 = x/2$, so the radius equals $x/4$ and the area of such a cross-section equals

$$A(x) = \pi(x/4)^2 = \frac{1}{16}\pi x^2.$$

The volume equals

$$V = \int_0^{12} A(x)dx = \int_0^{12} \frac{1}{16}\pi x^2 dx = \frac{1}{16}\pi\frac{x^3}{3}|_0^{12} = \frac{1}{16}\pi \cdot \frac{12^3}{3} =$$

$$\frac{12^3\pi}{3 \cdot 16} = \frac{1728\pi}{48} = 36\pi.$$

On the other hand, the volume of the circular cone with radius 3 and height 12 equals

$$V_{cone} = \frac{1}{3} \cdot 12 \cdot \pi \cdot 3^2 = 36\pi.$$

24. (10 points) Find the volume of the solid obtained by rotation of the region bounded by $y = x - x^2$ and $y = 0$ around the x-axis.

Solution: The graphs $y = x - x^2$ and $y = 0$ intersect at points where $x - x^2 = 0$, that is, $x = 0$ and $x = 1$. Now

$$V = \int_0^1 \pi(x - x^2)^2 dx = \pi\int_0^1 (x^2 - 2x^3 + x^4)dx = \pi(\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5})|_0^1 =$$

$$\pi(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}) = \pi\frac{20 - 2 \cdot 15 + 12}{60} = \frac{2\pi}{60} = \frac{\pi}{30}.$$

62. (10 points) The bob is obtained by the rotation of the curve $y = \frac{x}{12}\sqrt{36 - x^2}$ for $0 \leq x \leq 6$. Find the volume of the bob. If the brass density is $8.5 g/cm^3$, find the mass of the bob.

Solution: We have $f(x) = \frac{x}{12}\sqrt{36 - x^2}$, so

$$V = \int_0^6 \pi f(x)^2 dx = \pi \int_0^6 \frac{x^2}{12^2}(36 - x^2)dx = \frac{\pi}{144}\int_0^6 (36x^2 - x^4) =$$

$$\frac{\pi}{144}(36\frac{x^3}{3} - \frac{x^5}{5})|_0^6 = \frac{\pi}{144}(12 \cdot 6^3 - 6^5/5) = \pi(18 - 10.8) = 7.2\pi \simeq 22.6cm^3$$

Therefore the mass equals $22.6 \cdot 8.5 \simeq 192.3g$.