Section 6.1: 14. (10 points) A solid lies between the planes $x = 0$ and $x = 12$, and its cross-sections are circles whose diameters run from $y = x/2$ to $y = x$. Explain why the solid have the same volume as the circular cone with radius 3 and height 12.

Solution: The diameter of the cross-section equals $x - x/2 = x/2$, so the radius equals $x/4$ and the area of such a cross-section equals

$$A(x) = \pi \left(\frac{x}{4}\right)^2 = \frac{\pi}{16}x^2.$$  

The volume equals

$$V = \int_0^{12} A(x)\,dx = \int_0^{12} \frac{\pi}{16}x^2\,dx = \frac{1}{16}\pi \frac{x^3}{3}\bigg|_0^{12} = \frac{1}{16}\pi \cdot \frac{12^3}{3} = \frac{12^3\pi}{3 \cdot 16} = \frac{1728\pi}{48} = 36\pi.$$  

On the other hand, the volume of the circular cone with radius 3 and height 12 equals

$$V_{cone} = \frac{1}{3} \cdot 12 \cdot \pi \cdot 3^2 = 36\pi.$$  

24. (10 points) Find the volume of the solid obtained by rotation of the region bounded by $y = x - x^2$ and $y = 0$ around the $x$-axis.

Solution: The graphs $y = x - x^2$ and $y = 0$ intersect at points where $x - x^2 = 0$, that is, $x = 0$ and $x = 1$. Now

$$V = \int_0^1 \pi (x - x^2)^2\,dx = \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \pi \left(\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5}\right)|_0^1 = \pi \left(\frac{1}{3} - 2\frac{1}{4} + \frac{1}{5}\right) = \pi \frac{20 - 2 \cdot 15 + 12}{60} = \frac{2\pi}{60} = \frac{\pi}{30}.$$  

62. (10 points) The bob is obtained by the rotation of the curve $y = \frac{x}{12}\sqrt{36 - x^2}$ for $0 \leq x \leq 6$. Find the volume of the bob. If the brass density is $8.5\,g/cm^3$, find the mass of the bob.
Solution: We have \( f(x) = \frac{x}{12} \sqrt{36-x^2} \), so

\[
V = \int_0^6 \pi f(x)^2 \, dx = \pi \int_0^6 \frac{x^2}{12^2} (36-x^2) \, dx = \frac{\pi}{144} \int_0^6 (36x^2-x^4) = \\
\frac{\pi}{144} (36x^3/3-x^5/5) \bigg|_0^6 = \frac{\pi}{144} (12 \cdot 6^3 - 6^5/5) = \pi (18 - 10.8) = 7.2\pi \approx 22.6 \text{cm}^3
\]

Therefore the mass equals 22.6 \cdot 8.5 \approx 192.3g.