MAT 21B, Spring 2019
Solutions to homework 6

Section 7.2: Solve the differential equations:
12. (10 points) $\frac{dy}{dx} = 3x^2 e^{-y}$.

Solution: This is a separable equation, and we can write

$$\int e^y dy = \int 3x^2 dx,$$

so

$$e^y = x^3 + C,\ y = \ln(x^3 + C).$$

18. (10 points) $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}}$.

Solution: Let us simplify the right hand side first:

$$\frac{e^{2x-y}}{e^{x+y}} = e^{2x-y} \cdot e^{-x-y} = e^x e^{-2y}.$$

Now $dy/dx = e^x e^{-2y}$, so

$$\int e^{2y} dy = \int e^x dx,$$

$$\frac{1}{2} e^{2y} = e^x + C,\ e^{2y} = 2(e^x + C),\ 2y = \ln(2(e^x + C))$$

and

$$y(x) = \frac{1}{2} \ln(2(e^x + C)).$$

32.(10 points) The concentration of the antibiotic in blood is modeled by the differential equation $y' = r - ky$.

(a) If $y(0) = y_0$, find $y(t)$ at any time t.

(b) Assume that $y_0 < r/k$ and find limit $\lim_{t\to\infty} y(t)$. Sketch the solution curve for $y(t)$.

Solution: We have $\frac{dy}{dt} = r - ky$, so

$$\int \frac{dy}{r - ky} = \int dt,$$

$$\frac{-1}{k}\ln|r-ky| = t + C,\ \ln|r-ky| = -kt - kC$$
$$r - ky = \pm e^{-kt-kC} = Ae^{-kt} \quad (A = \pm e^{-kC}).$$

Therefore $ky = r - Ae^{-kt}$ and

$$y(t) = \frac{1}{k}(r - Ae^{-kt}).$$

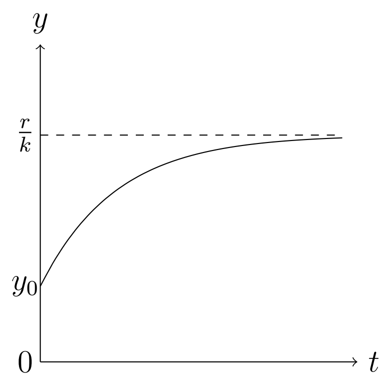
Now we need to find the constant A from the initial condition. We have

$$ky(0) = r - A, A = r - ky_0,$$

so

$$y(t) = \frac{1}{k}(r - (r - ky_0)e^{-kt}) = \frac{r}{k} - \frac{r}{k}e^{-kt} + y_0e^{-kt}.$$

(b) Since $\lim_{t\to\infty} e^{-kt} = 0$, we have $\lim_{t\to\infty} y(t) = \frac{r}{k}$.



36.(10 points) The price $p(x)$for x units satisfies the differential equation

$$\frac{dp}{dx} = \frac{-p}{100},\ p(100) = 20.09$$

(a) Find the formula for $p(x)$.
(b) Find $p(10)$ and $p(90)$.
(c) Show that the revenue $r(x) = xp(x)$ has maximum at $x = 100$.
(d) Graph the revenue $r(x)$.

Solution: (a) We have $p(x) = Ae^{-x/100}$, so $p(100) = Ae^{-100/100} = Ae^{-1}$. We have

$$Ae^{-1} = 20.09,\ A = 20.09e,$$

so

$$p(x) = 20.09e \cdot e^{-x/100} = 20.09e^{1-\frac{x}{100}} = 20.09e^{\frac{100-x}{100}}.$$

(b) If $x = 10$ we have $p(x) = 20.09e^{0.9} \approx 49.41$. If $x = 90$ then $p(x) = 20.09e^{0.1} \approx 22.20$.

(c) We have $r(x) = Axe^{-x/100}$, so

$$r'(x) = Ae^{-x/100} + Ax \cdot \frac{-1/100}{e}^{-x/100} = A(1 - \frac{x}{100})e^{-x/100}.$$

The derivative is positive if $1 - \frac{x}{100} > 0$, so $100 - x > 0$, $x < 100$. Therefore the function $r(x)$ increases for $x < 100$ and decreases for $x > 100$, so it has a maximum at $x = 100$.

(d)