Lecture 20 (2/26)

Sunday, February 25, 2024 8:46 PM

Linearization and tangent spaces

$f(x_1 \dots x_n)$ = polynomial $\Rightarrow$ can define $\frac{\partial f}{\partial x_i}$ by usual formulas

Works over any field! With some warnings.

Ex Suppose char $\mathbb{K} = 0$, and $\frac{\partial f}{\partial x_i} = 0$ for all i.
Then f is a constant.

Pf: $f = f_0(x_1 \dots x_{n-1}) + f_1(x_1 \dots x_{n-1}) \cdot x_n + \dots + f_d(x_1 \dots x_{n-1}) x_n^d$

$$\frac{\partial f}{\partial x_n} = f_1(x_1 \dots x_{n-1}) + \dots + d\, f_d(x_1 \dots x_{n-1})\, x_n^{d-1}$$

If $\frac{\partial f}{\partial x_n} = 0$ then $f_i(x_1 \dots x_{n-1}) = 0$ for all $i > 0$
and $f = f_0(x_1 \dots x_{n-1})$, now proceed by induction. $\boxtimes$

Ex $f = x_1^p + \dots + x_n^p$, and char $\mathbb{K} = p$. Then $\frac{\partial f}{\partial x_i} = 0$ for all i

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Def Given a polynomial $f(x_1 \dots x_n)$ and a point $p = (p_1 \dots p_n)$, we define the <u>linearization</u> of f at p as

$$L_p(f; b_1 \dots b_n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot b_i$$

<u>Motivation</u>: In calculus,

$$f(p_1 + b_1, \dots p_n + b_n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot b_i + \dots$$

$f(p_1+b_1,\dots,p_n+b_n) = f(p) + \sum_{i=1} \frac{\partial f}{\partial x_i}(p)\cdot b_i + \dots$

__Rmk__ Assume $p_1 = \dots = p_n = 0$. Then $\frac{\partial f}{\partial x_i}(0\dots 0) =$ coef at x_i in f

If $f = a_0 + \sum a_i x_i + \dots$ then $\boxed{L_0(f;b) = a_0 + \sum a_i b_i}$.

__Def__ Suppose $X = Z(I) \subset \mathbb{A}^n$ and $p \in X$.
Then __Zariski tangent space__ $T_p X$ is defined as

$$T_p X = \{(b_1 \dots b_n) : \underbrace{L_p(f; b_1 \dots b_n) = 0}_{(*)} \text{ for all } f \in I\}.$$

__Note__: For $p \in X$, $f(p) = 0$, so $L_p(f; b_1 \dots b_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\cdot b_i$

__lemma__ 1) It is sufficient to check $(*)$ for generators of I.

2) $T_p X$ is a vector space.

__Proof__: 1) Suppose $L_p(f; \bar{b}) = 0$, then for all g.

$$L_p(fg; \bar{b}) = \sum \frac{\partial(fg)}{\partial x_i}(p)\cdot b_i =$$

$$= g(p)\cdot \underbrace{\sum \frac{\partial f}{\partial x_i}(p)\cdot b_i}_{=0} + \underbrace{f(p)}_{=0}\cdot \sum \frac{\partial g}{\partial x_i}(p)\cdot b_i$$

2) $L_p(f; \bar{b} + \bar{c}) = L_p(f; \bar{b}) + L_p(f; \bar{c})$
So if $\bar{b}, \bar{c} \in T_p X$ then $\bar{b} + \bar{c} \in T_p X$.

__Ex__ $X = \{x^2 = y^3\} \subset \mathbb{A}^2$

$I = (x^2 - y^3) = (f)$ $\qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = -3y$.

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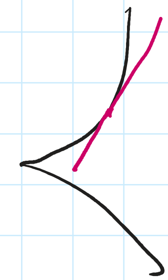
$p=(x,y)$ $\quad T_pX = \{ (b_1, b_2) : 2x \cdot b_1 - 2y \cdot b_2 = 0 \}$



If $(x,y) \neq (0,0)$ then T_pX is a line

If $(x,y) = (0,0)$ then $T_pX = \mathbb{A}^2$!

<u>Ex</u> $X = \{x_1^p + \dots + x_n^p = 0\}$, char $\mathbb{K} = p$

Then $\frac{\partial f}{\partial x_i} = 0$, so $\dim T_q X = n$ for all q

while $\dim X = n-1$ (one equation in $\mathbb{A}^n$).

<u>Lemma</u> $X = \{ f_1 = \dots = f_k = 0\}$ in $\mathbb{A}^n$.

Define Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

Then $(b_1, \dots, b_n) \in T_pX \iff J \cdot \overline{b} = 0$

and $\boxed{\dim T_pX = n - \operatorname{rank} J}$

<u>Thm</u> Define $X_k = \{ p \in X : \dim T_pX \geq k \}$. Then

X_k is closed in X. ("$\dim T_pX$ is semicontinuous").

$\lim_{t \to 0} x_t = x_0 \quad \dim T_{x_0} X \geq \dim T_{x_t} X$

<u>Proof</u>: $\dim T_pX \geq k \iff$

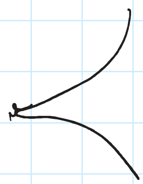
$\iff \operatorname{rank} J \leq n-k$

$\Longleftrightarrow$ all $(n-k+1)\times(n-k+1)$ minors of J vanish.

J is a matrix of polynomials, so any minor is a polynomial in $x_1 \dots x_n$ as well.

So we get a Zariski closed condition. ∎

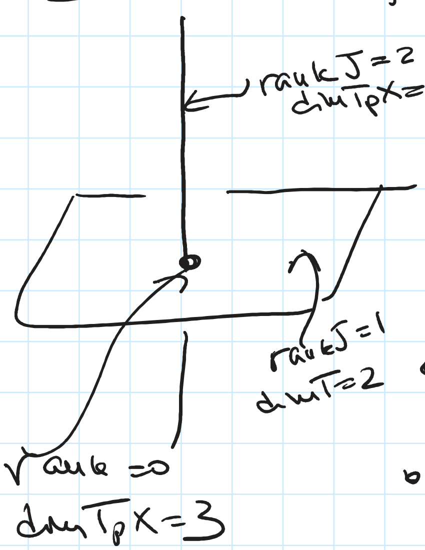
Ex



$X = X_1 \qquad X_2 = \{(0,0)\}$.

$X = \{x^2 = y^3\}$.

Ex $X = \{xz = yz = 0\} \subset \mathbb{A}^3 \qquad I = (xz, yz)$



$$J = \begin{pmatrix} z & 0 & x \\ 0 & z & y \end{pmatrix}$$ What is $\operatorname{rank}(J)$?

- $z \neq 0 \Rightarrow \operatorname{rank}(J) = 2$, $\dim T_pX = 1$
- $z = 0$, x or $y \neq 0 \Rightarrow \operatorname{rank}(J) = 1$, $\dim T_pX = 2$
- $z = x = y = 0 \Rightarrow \operatorname{rank}(J) = 0$, $\dim T_pX = 3$

Sometimes it is helpful to have two abstract characterizations of T_pX.

Thm: Vectors $(b_1 \dots b_n) \in T_pX$ are in bijection with homomorphisms $A(X) \xrightarrow{\phi} \mathbb{K}[\varepsilon] \xrightarrow{\varepsilon = 0} \mathbb{K} = \mathbb{K}[\varepsilon]$

with homomorphisms

$$\begin{tikzcd} A(X) \arrow[r, "\phi"] \arrow[rr, bend right, "\phi_0"'] & \frac{\mathbb{K}[\varepsilon]}{\varepsilon^2} \arrow[r, "\varepsilon=0"] & \mathbb{K} = \frac{\mathbb{K}[\varepsilon]}{\varepsilon} \end{tikzcd}$$

$\phi : A(X) \longrightarrow \frac{\mathbb{K}[\varepsilon]}{\varepsilon^2}$

Such that $\phi_0(f) = f(p)$.

Pf: Let us unpack the definition of such ϕ.

We have $A(X) = \frac{\mathbb{K}[x_1 \dots x_n]}{I(X)} \xrightarrow{\phi} \frac{\mathbb{K}[\varepsilon]}{\varepsilon^2}$

This is determined by $x_i \longrightarrow p_i + b_i \varepsilon$

$$f(x_1 \dots x_n) \longmapsto f(p_1 + b_1\varepsilon, \dots, p_n + b_n\varepsilon)$$
$$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{L_p(f; \varepsilon b_1 \dots \varepsilon b_n)} \ \underline{\text{mod } \varepsilon^2}$$

This is well defined if $\phi(I(X)) = 0$

$$\iff \begin{cases} f(p) = 0 \\ L_p(f; \varepsilon b_1 \dots b_n) = \varepsilon L_p(f; b_1 \dots b_n) = 0 \end{cases}$$

$\blacksquare$

Rmk Sometimes this thm is abbreviated by saying that $TX \longleftrightarrow \text{Maps}(\text{Spec}\,\frac{\mathbb{K}[\varepsilon]}{\varepsilon^2}, X)$.

Here $\text{Spec}\,\frac{\mathbb{K}[\varepsilon]}{\varepsilon^2} = Y$ is an imaginary "space" (actually, a scheme) such that $A(Y) = \frac{\mathbb{K}[\varepsilon]}{\varepsilon^2}$. Then $\text{Maps}(Y, X) \longleftrightarrow \text{Hom}(A(X), A(Y))$.

$h(t) = \frac{1}{t^2}$. Then maps [illegible]