Lecture 18 X = alg. variety

Def $\dim X$ = length of maximal chain of irreducible closed subsets of X: $\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$

Easy $\dim \mathbb{A}^1 = 1$ Closed subsets are $\emptyset, \mathbb{A}^1$, finite sets.

Max. chain = $\{p\} \subset \mathbb{A}^1$.

Hard $\dim \mathbb{A}^n = n$. Proof later, need a lot of comm. algebra

Lemma $X = X_1 \cup \dots \cup X_k$ irred. components

Then $\dim X = \max (\dim X_i)$

Recall Z = closed, irreducible $\Rightarrow Z = Z \cap X_i$ for some i

So $Z \subset X_i$

$Z_0 \subset Z_1 \subset \dots \subset Z_n \Rightarrow Z_n \subset X_i \Rightarrow \dim X \leq \dim X_i$.

Def A = algebra, then (Krull) dimension of A is the length of maximal chain of prime ideals in A:

$p_1 \subsetneq p_2 \subsetneq \dots \subsetneq p_n \subsetneq A$.

Def A = algebra over $\mathbb{K}$, $a_1 \dots a_n$ are algebraically independent if $f(a_1 \dots a_n)$ for a polynomial f implies $f = 0$.

Def The transcendence degree of A

$$\operatorname{trdeg}(A) = \max\{n : \exists a_1 \dots a_n \text{ alg. independent in } A\}.$$

Note $\operatorname{trdeg} A \geq n \iff \exists\, a_1 \ldots a_n$ alg-independent

$\iff$ there exists an injective homomorphism

$\mathbb{K}[x_1 \ldots x_n] \hookrightarrow A \qquad (x_i \mapsto a_i)$.

Note $A = A(X)$ for affine alg. set X, then

inj. homomorph $\mathbb{K}[x_1 - x_n] \hookrightarrow A(X)$

$\iff$ dominant map $X \longrightarrow \mathbb{A}^n$

Cor $\operatorname{trdeg} A(X) \geq n \iff$ there exists a dominant map $X \to \mathbb{A}^n$.

Lemma 1 Assume A = integral domain (no zero divisors) and $I \subset A$ nonzero ideal. Then

$$\operatorname{trdeg}(A/I) < \operatorname{trdeg}(A).$$

Pf Assume $\operatorname{trdeg}(A) = n$, need to prove $\bar{a}_1 - \bar{a}_n \in A/I$ are alg. dependent. Choose $a_i \in A$ project to $\bar{a}_i$, and $a \in I$. Then $a_1 - a_n, a$ alg. dependent in $A \Rightarrow g(a_1 - a_n, a) = 0$

$$\sum_{k \geq 0} a^k g_k(a_1 - - a_n) = 0$$

$g_0 = 0$, can divide by a (since A is a domain) $\Rightarrow$ assume $g_0 \neq 0$.

$$g_0(a_1 - a_n) = -\sum_{k>0} a^k g_k(a - a_n) \in I$$

$\Rightarrow g_0(\bar{a}_1, - \bar{a}_n) = 0 \mod I$. $\square$

Lemma 2 Assume A is a domain, $\operatorname{trdeg}(A)=0$.
Then A is a field.

Pf Similar to lemma 1, $g(a)=0$ for $a\in A$

$$\sum a^k g_k = 0 \quad g_0 \neq 0 \Rightarrow \qquad g_i \in k$$

$$1 = \frac{1}{g_0} a\cdot\left(-\sum_{k\geq 1} g_k a^{k-1}\right) \Rightarrow a \text{ is invertible.}$$

Thm For any A $\quad \dim A \leq \operatorname{trdeg} A$.

Pf Suppose $\operatorname{trdeg} A = n$, run induction on n

Base ($n=0$) If $\operatorname{trdeg} A = 0$ and $\mathfrak{p}_0 \subset A$ prime
then $A/\mathfrak{p}_0$ is a domain $\operatorname{trdeg}(A/\mathfrak{p}_0)=0$
$\Rightarrow$ by Lemma 2 $\quad A/\mathfrak{p}_0$ is a field $\Rightarrow$ no ideals in $A/\mathfrak{p}_0$
and $\dim A \leq 0$.

Step Assume we proved for $\operatorname{trdeg} \leq n-1$
Given a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subseteq \dots \subsetneq \mathfrak{p}_k \subset A$$

$$0 \subsetneq \mathfrak{p}_1/\mathfrak{p}_0 \subsetneq \mathfrak{p}_2/\mathfrak{p}_0 \subset \dots \subseteq \mathfrak{p}_k/\mathfrak{p}_0 \subseteq \underbrace{A/\mathfrak{p}_0}_{\text{domain}}$$

By Lemma 1 $\quad \operatorname{trdeg}(A/\mathfrak{p}_1) = \operatorname{trdeg}\left(\frac{A/\mathfrak{p}_0}{\mathfrak{p}_1/\mathfrak{p}_0}\right) < \operatorname{trdeg} A/\mathfrak{p}_0 \leq n$

Therefore $\quad k-1 \leq \dim(A/\mathfrak{p}_1) \underset{\text{assumption of induction}}{\leq} \dim \quad \operatorname{trdeg}(A/\mathfrak{p}_1) < n$

and $k \leq n$. $\square$

<u>Thm</u> a) tr deg $\mathbb{K}[x_1 \ldots x_n] = n$

b) $\dim \mathbb{A}^n = \dim \mathbb{K}[x_1 \ldots x_n] = n$.

<u>Pf</u> (a) $x_1 \ldots x_n$ alg. independent $\Rightarrow$ tr deg $\mathbb{K}[x_1 \ldots x_n] \geq n$.

Assume $f_1 \ldots f_{n+1}$ are polynomials, $d = \max(\deg f_i)$.

There are $\sim \frac{k^{n+1}}{(n+1)!}$ products $f_1^{\alpha_1} \ldots f_{n+1}^{\alpha_{n+1}}$ $\alpha_i \leq k$.

all of degree $< dk$.

On the other hand, there are $\sim \frac{(dk)^n}{n!}$ monomials of deg $\leq dk$

For $k \gg 0$ $\frac{k^{n+1}}{(n+1)!} \gg \frac{(dk)^n}{n!}$ and ~~the result follows so~~

$f_1^{\alpha_1} \ldots f_{n+1}^{\alpha_{n+1}}$ are dependent for different α.

(b) By Thm $\dim \mathbb{K}[x_1 \ldots x_n] \leq$ tr deg $\mathbb{K}[x_1 \ldots x_n] = n$.

On the other hand $(0) \subset (x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1 \ldots x_n)$

chain of prime ideals, so $\dim \mathbb{K}[x_1 \ldots x_n] \geq n$. $\square$