Lecture 19 Recap $\dim A \le \operatorname{trdeg} A$

$\dim \mathbb{A}^n = \operatorname{trdeg} \mathbb{K}[x_1 \ldots x_n] = n$

Thm Suppose X = affine alg. set in $\mathbb{A}^n$

Then $\dim X = \dim A(X) = \operatorname{trdeg} A(X)$

Proof Skip.

Cor Suppose $\varphi: X \to Y$ dominant, then $\dim X \ge \dim Y$

Pf φ dominant $\Leftrightarrow$ $\varphi^*: A(Y) \to A(X)$ injective

$a_1 \ldots a_n$ alg. indep in $A(Y)$ $\Rightarrow$ $\varphi^*(a_1) \ldots \varphi^*(a_n)$ alg. indep in $A(X)$

So $\operatorname{trdeg} A(X) \ge \operatorname{trdeg} A(Y) \Rightarrow \dim X \ge \dim Y$.

Cor X = irreducible, closed $\overset{\text{proper}}{\text{subset}}$ in Y, Y irreducible.

Then $\dim X < \dim Y$

Pf 1: $X_0 \subsetneq \ldots \subsetneq X_n \subset X$ chain extends to

$X_0 \subset \ldots \subsetneq X_n \subset Y$ so $\dim Y \ge n+1$.

Pf 2 $A(Y)$ is a domain since Y is irreducible

$\Rightarrow A(X) = \dfrac{A(Y)}{I}$, ~~dim~~ $\operatorname{trdeg} A(X) < \operatorname{trdeg} A(Y)$.

by last lecture.

Ex $X \subset \mathbb{A}^n$ closed, proper alg. set $\Rightarrow \dim X < n$.

Fact (w/o proof) If A is a domain then

$$\operatorname{trdeg} A = \operatorname{trdeg} \operatorname{Quot}(A) \leftarrow \text{field of fractions.}$$

Lemma X irreducible, $U \subseteq X$ open nonempty $\Rightarrow \dim U = \dim X$

Pf: $D(f) \subset U \subset X$ $\qquad \underset{\operatorname{trdeg} A(X)[f^{-1}]}{\underset{\parallel}{\dim D(f)}} \leq \dim U \leq \underset{\operatorname{trdeg} A(X)}{\underset{\parallel}{\dim X}}$

But ~~Quot Ba~~ $\underset{\operatorname{trdeg} \operatorname{Quot}(A(X))}{\underset{\parallel}{\operatorname{trdeg} A(X)[f^{-1}]}} = \underset{\parallel}{\operatorname{trdeg} A(X)} = \dim X$

and we are done

Fact (w/o proof) $\dim (X \times Y) = \dim X + \dim Y$

Thm (Krull principal ideal theorem)

X irreducible, $f \in A(X)$ and $\{f=0\} \neq X, \emptyset$

Then all irreducible components of $\{f=0\}$ have dimension exactly $\dim X - 1$.

Equivalently $\operatorname{trdeg} A/(f) = \operatorname{trdeg} A - 1$.

(w/o proof, see Clader–Ross chapter 6).

Cor X irreducible, $Y = \{f_1 = \dots = f_k = 0\}$

Then all irreducible components of Y have dimension $\dim (Y_i) \geq \dim X - k$.

Pf Induction on k. $Z = \{f_1 = \dots = f_{k-1}\} = \bigcup Z_i$ irred components.

$\dim Z_i \geq \dim X - k + 1$

$Y = \{f_k = 0\} \cap Z = \bigcup (\{f_k = 0\} \cap Z_i)$

Case 1: $\{f_k = 0\} \cap Z_i = \emptyset$ OK

Case 2: $\{f_k = 0\} \cap Z_i = Z_i \Rightarrow \dim \geq \dim X - k + 1$

Case 3: $\{f_k = 0\} \cap Z_i \neq \emptyset, Z_i \Rightarrow$ by Thm all irred components of $\{f_k = 0\} \cap Z_i$ have $\dim = \dim Z_i - 1 \geq \dim X - k$.

Def $Y = \{f_1 = \dots = f_k = 0\} \subset X$ is a complete intersection if for all irred. components $\dim Y_i = \dim X - k$.

Ex: Twisted cubic $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$[x_0 : x_1] \longrightarrow [y_0 : y_1 : y_2 : y_3] = [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$

HW: C cut out by 3 equations $y_0 y_2 = y_1^2$, $y_1 y_3 = y_2^2$, $y_0 y_3 = y_1 y_2$

$\mathbb{P}^1 \simeq C \Rightarrow \dim C = 1$

$\qquad\qquad\qquad\qquad \vee|$

$\dim \mathbb{P}^3 - \#\text{ eqns} = 3 - 3 = 0.$

In more details:

① $\{y_0 y_3 = y_1 y_2\}$ is nondegenerate quadric $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ irreducible, $\dim = 3 - 1 = 2$.

② $\{y_0 y_3 = y_1 y_2,\ y_0 y_2 = y_1^2\} = Z$

- If $y_0 = 0$ then $y_1 = 0$, we get line $[0:0:y_2:y_3] \simeq \mathbb{P}^1$
- If $y_0 \neq 0$ then $y_2 = \frac{y_1^2}{y_0}$, $y_3 = \frac{y_1 y_2}{y_0} = \frac{y_1^3}{y_0^2}$

$[y_0 : y_1 : y_2 : y_3] = [y_0 : y_1 : \frac{y_1^2}{y_0} : \frac{y_1^3}{y_0^2}] = [1 : \frac{y_1}{y_0} : \left(\frac{y_1}{y_0}\right)^2 : \left(\frac{y_1}{y_0}\right)^3]$

this is our C (in chart $y_0 \neq 0$)

So $Z = \mathbb{P}^1 \cup C$ dim (both components) $= 1 = 3-2$ ✓

③ $\{y_0 y_3 = y_1 y_2,\ y_0 y_2 = y_1^2,\ y_1 y_3 = y_2^2\} = Z \cap \{y_1 y_3 = y_2^2\}$

holds in C!

$\mathbb{P}^1 \cap \{y_1 y_3 = y_2^2\} = [0:0:0:1]$ pt
$\Rightarrow y_2 = 0$

$C \cap \{y_1 y_3 = y_2^2\} = C$.

<u>Thm</u> $X, Y \subset \mathbb{A}^n$ irreducible, closed

Then all components of $X \cap Y$ have $\dim \geq \dim X + \dim Y - n$

<u>Pf</u> Consider $X \times Y \subset \mathbb{A}^{2n}$, $Z = X \times Y \cap \underbrace{\{x_1 = y_1, \dots, x_n = y_n\}}_{\text{diagonal}}$

$\dim X \times Y = \dim X + \dim Y$, n eqn

$\dim Z_i \geq \dim X + \dim Y - n$

Z_i = components of Z.

$Z \simeq X \cap Y$.
