Lecture 20 | Linearization and tangent spaces

Let's do multivariable calculus over any field $\mathbb{K}$!

- Given a polynomial $x_1^{a_1} \cdots x_n^{a_n}$, define
  $$\frac{\partial}{\partial x_i}(x_1^{a_1} \cdots x_n^{a_n}) = a_i\, x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_n^{a_n}$$, extend by linearity.
  $$\frac{\partial}{\partial x_i} : \mathbb{K}[x_1 \ldots x_n] \longrightarrow \mathbb{K}[x_1 \ldots x_n]$$
- $\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}$ (product rule)
  Also Quotient Rule, Chain Rule...
- $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$.

Def Given a polynomial $f(x_1 \ldots x_n)$ and a point $p = (p_1 \ldots p_n)$ we define the linearization of f at p as
$$L_p(f; b_1 \ldots b_n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\, b_i$$

Def Suppose $X = Z(I) \subset \mathbb{A}^n$ and $p \in X$.
The Zariski tangent space $T_p X$ is defined as
$$T_p X = \{ (b_1 \ldots b_n) : \underset{(*)}{L_p(f; b_1 \ldots b_n) = 0} \text{ for all } f \in I \}.$$

Note for $p \in X$, $f(p) = 0$, so $L_p(f; b_1 \ldots b_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot b_i$
This is linear in $\vec{b}$, so $T_p X$ is a vector space.

Lemma It is sufficient to check $(*)$ for generators of I.

Pf Suppose $L_p(f; \vec{b}) = 0$ then

$$L_p(fg;b) = \sum_{i=1}^{n} \frac{\partial (fg)}{\partial x_i}(p)\, b_i = \overbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\cdot b_i \cdot g(p)}^{0} + \sum_{i=1}^{n} \underbrace{f(p)}_{0} \cdot \frac{\partial g}{\partial x_i}(p)\cdot b_i$$

So $(*)$ holds for fg. □

Ex $X = \{x^2+y^2+z^2=1\} \subset \mathbb{A}^3$

$f = x^2+y^2+z^2-1$ $\qquad \frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = 2y, \ \frac{\partial f}{\partial z} = 2z$

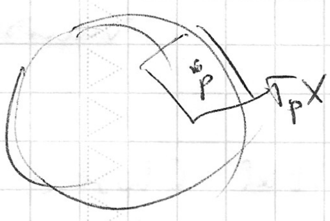
$p = (p_1, p_2, p_3)$

$L_p(f) = 2p_1\cdot b_1 + 2p_2 \cdot b_2 + 2p_3 \cdot b_3$

Case 1 $\operatorname{char} \mathbb{K} = 2$ Then $L_p(f) \equiv 0$ and $\dim T_pX = 3$

Case 2 $\operatorname{char} \mathbb{K} \neq 2$ If at least one $p_1, p_2, p_3 \neq 0$ then

$L_p(f;b) = 0 \iff$ hyperplane in $\mathbb{A}^3$ $\dim T_pX = 2$



If $p_1 = p_2 = p_3$ then $p_1^2+p_2^2+p_3^2 = 0$, contradiction.

So $\dim T_pX = 2$ at all points p.

Ex $X = \{x^2 = y^3\} \subset \mathbb{A}^2$ $\quad f = x^2 - y^3$

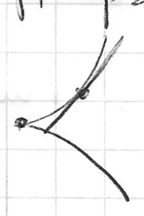
$\frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = -3y^2$ $\quad$ Assume $\operatorname{char} \mathbb{K} \neq 2, 3$

$L_p(b_1, b_2) = 2p_1 b_1 - 3p_2^2 b_2$ $\qquad p = (p_1, p_2)$

Case 1: p_1 or $p_2 \neq 0$ then $\dim T_pX = 1$

Case 2: $p_1 = p_2 = 0$ then $\dim T_pX = 2$

singular point!



Lemma $X = \{f_1 = \dots = f_k = 0\}$ in $\mathbb{A}^n$

Define Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

Then $(b_1, \dots, b_n) \in T_pX \iff J \cdot \vec{b} = 0$

So $T_pX = \operatorname{Ker} J$, $\boxed{\dim T_pX = n - \operatorname{rank} J}$

Thm Define $X_k = \{p \in X : \dim T_pX \geq k\}$ Then X_k is closed in X.

("$\dim T_pX$ is semicontinuous")

Pf: $\dim T_pX \geq k \iff$

$\iff \operatorname{rank} J \leq n-k \iff$ all $(n-k+1) \times (n-k+1)$ minors of J vanish.

This is a system of polynomial equations (all entries of J are polynomials) $\Rightarrow$ Zariski closed

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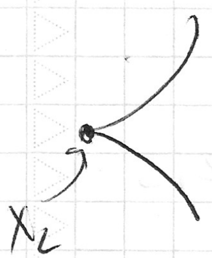


$X = X_1 = \{x^2 = y^3\}$

$X_2 = \{(0,0)\}$

Ex $X = \{xz = yz = 0\} \subset \mathbb{A}^3$

$I = (xz, yz)$

$$J = \begin{pmatrix} z & 0 & x \\ 0 & z & y \end{pmatrix}$$ what is $\operatorname{rank}(J) = ?$

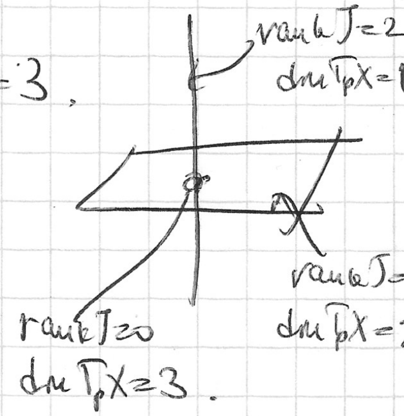
- $z \neq 0 \Rightarrow \text{rank}(J) = 2$, $\dim T_pX = 1$

- $z = 0$ we get $\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} \Rightarrow \text{rank}(J) = 1$

  x or $y \neq 0$

  $\dim T_pX = 2$

- $z = x = y = 0 \Rightarrow \text{rank}(J) = 0$, $\dim T_pX = 3$.



Ex Twisted cubic

$\mathbb{P}^3 \ni [y_0 : y_1 : y_2 : y_3]$ equation $y_0 y_2 = y_1^2$

$y_0 y_3 = y_1 y_2$

$y_1 y_3 = y_2^2$

Focus on affine chart $y_0 \neq 0$ then $\begin{cases} y_2 - y_1^2 = 0 \\ y_3 - y_1 y_2 = 0 \\ y_1 y_3 - y_2^2 = 0 \end{cases}$

so $y_0 = 1$

$$J = \begin{pmatrix} -2y_1 & 1 & 0 \\ -y_2 & -y_1 & 1 \\ y_3 & -2y_2 & y_1 \end{pmatrix} \qquad \text{Rank}(J) = ?$$

Idea: substitute $y_2 = y_1^2$, $y_3 = y_1^3$

$$J = \begin{pmatrix} -2y_1 & 1 & 0 \\ -y_1^2 & -y_1 & 1 \\ y_1^3 & -2y_1^2 & y_1 \end{pmatrix}$$

III row $= y_1 \cdot$ II row $- y_1^2 \cdot$ I row

$\Rightarrow \text{rank } J = 2$

$\dim T_pX = 1$ (over any field).