Lecture 21 | Recall $L_p(f; b_1 \dots b_n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) b_i$

for $p \in X$ $T_p X = \{ (b_1 \dots b_n) : L_p(f; b_1 \dots b_n) = 0 \ \forall f \in I(X) \}$

If $X = \{ f_1 = \dots = f_k = 0 \}$ then $J = \left( \frac{\partial f_i}{\partial x_j} \right)$ Jacobian matrix

$\boxed{T_p X = \ker J, \ \dim T_p X = n - \operatorname{rank} J}$

**Thm** Let $m_p = (x_1 - p_1, \dots x_n - p_n) \subset A(X)$ be the maximal ideal in $A(X)$ corresponding to $p \in X$.

Then $(T_p X)^* \simeq m_p / m_p^2$

Rmk This definition is intrinsic in terms of $A(X)$.

Pf ① Given $\bar{b} = (b_1 \dots b_n) \in T_p X$, define a $\mathbb{K}$-linear functional $\ell_b : m_p \longrightarrow \mathbb{K}$

$$\ell_b(f) = L_p(f; b_1 \dots b_n)$$

$m_p \subset A(X) = \frac{\mathbb{K}[x_1 \dots x_n]}{I(X)}$ $\quad$ $\ell_b(f) = 0$ for $f \in I(X)$

so $\ell_b$ is well defined in $m_p$ (in fact, in $A(X)$).

If $f, g \in m_p$ then $\ell_b(fg) = \ell_b(f) g(p) + \ell_b(g) f(p) = 0$

since $f(p) = g(p) = 0$. So $\ell_b(m^2) = 0$ and we get

$$\ell_b : m_p / m_p^2 \longrightarrow \mathbb{K}$$

$T_p X \longrightarrow (m_p / m_p^2)^*$ We need to prove it is an isomorphism.

WLOG $p=(0 \dots 0)$, $X=\{f_1 = \dots = f_k = 0\}$

$$m_p / m_p^2 = \frac{(x_1 \dots x_n)}{(f_1 \dots f_k) + (x_1 \dots x_n)^2} = \frac{\mathrm{Span}(x_1 \dots x_n)}{(\text{linear part of } f_i)}$$

$T_pX = \{ b \in \mathrm{Span}(x_1 \dots x_n)^* : b(\text{linear part of } f_i = 0) \}$

and the result follows. $\square$

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Cor: If $\varphi : X \xrightarrow{\sim} Y$ then we get $T_pX \simeq T_{\varphi(p)}Y$

since $\varphi^* : A(Y) \to A(X)$ is a ring isomorphism.

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Lemma $T_pX$ is local, that is, if $U \subset X$ open, $p \in U$

then $T_pU = T_pX$.

Proof Consider $U = D(g) = \{g \neq 0\}$ $\quad g(p) \neq 0$, $f(p) = 0$

$$L_p\left(\frac{f}{g}, b\right) = \frac{g(p) L_p(f; b) - f(p) L_p(g; b)}{g(p)^2} =$$

$$= \frac{1}{g(p)} L_p(f; b)$$

So $L_p\left(\frac{f}{g}; b\right) = 0$ iff $L_p(f; b) = 0$, same for $g^k$.

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Thm Suppose $X$ is irreducible affine, then

$\dim T_pX \geq \dim X$.

Lemma If $\dim T_pX = 0$ then $\dim X = 0$

Pf: If $\dim X = 0$ and $X$ is irreducible then $X = \text{point}$.

Assume $\dim T_p X = 0$ then $m_p / m_p^2 = 0$, $m_p = m_p^2$

Let $F_1 \dots F_k$ = generators of $m_p$

we can write $F_i = \sum G_{ij} F_j$ for $G_{ij} \in m_p$
(since $F_i \in m_p^2$)

then $(I - G)\begin{pmatrix} F_1 \\ \vdots \\ \vdots \\ F_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $(*)$

Since $G_{ij} \in m_p$ we have $G_{ij}(p) = 0 \Rightarrow \det(I - G(p)) = 1$

$\Rightarrow U = \{\det(I - G(x)) \neq 0\}$ nonempty open
contains $p$

By $(*)$ we get $F_1 = \dots = F_k = 0$ on $U \Rightarrow F_1 = \dots = F_k = 0$ everywhere
($U$ is dense)

Contradiction.

$\square$

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<u>Rmk</u> This is a variant of "Nakayama's Lemma"

If $R$ is a <u>local</u> ring with maximal ideal $m$

and $H_1 \dots H_s \in m$ and span $m/m^2$

then $H_1 \dots H_s$ generate $m$. ~~as a ring~~

<u>Pf</u>: Define $\tilde{R} = R/(H_1 \dots H_s)$ then $m_{\tilde{R}} = m_R/(H_1 \dots H_s)$

and $m_{\tilde{R}}/m_{\tilde{R}}^2 = 0$. Similar to Lemma we get $m_{\tilde{R}} = 0$

$\Rightarrow m_R = (H_1 \dots H_s)$

Proof of Thm Induction on $n = \dim(X)$

① $n=0 \Rightarrow \dim T_pX \geq 0$ OK

② $n>0$ then by lemma $\dim T_pX > 0$

Choose a vector $\bar{b} \neq 0$, $b \in T_pX$

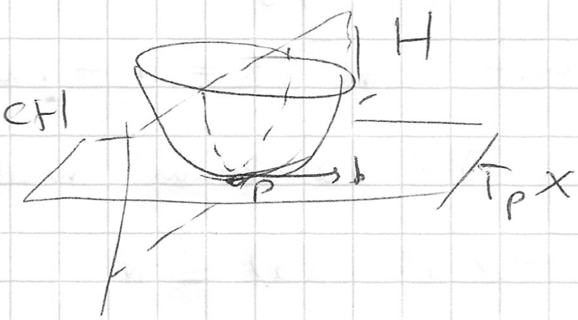
Choose a hyperplane $H \subset \mathbb{A}^N$ through $p$ such that $b \notin H$

$Y = X \cap H$

• $Y \neq \emptyset$ since $p \in Y$

• $Y \neq X$ indeed, if $X \subset H$ then $T_pX \subset H \Rightarrow b \in H$

Contradiction.




$\Rightarrow \dim(\text{all components of } Y) = \dim X - 1$

$\dim T_pX \geq \dim T_pY + \underset{b}{1} \geq (\dim X - 1) + 1 = \dim X$

assumption of induction*

and we are done.

* $Y = \bigcup Y_i$ $\dim Y_i = \dim X - 1$

$\dim T_pY \geq \dim T_pY_i \geq \dim Y_i = \dim X - 1$.