Lecture 22 | Recap: $\dim T_p X \geq \dim X$

Def Assume X is irreducible. A point p is smooth if $\dim T_p X = \dim X$, and singular if $\dim T_p X > \dim X$.

$$\operatorname{Sing} X = \{\text{singular points on } X\}.$$

Lemma $\{p : \dim T_p X \geq j\}$ is closed in X

Pf Recall that $\dim T_p X = N - \operatorname{rank}(J)$

$X \subset \mathbb{A}^N$ $\qquad J = $ Jacobian matrix

$\dim T_p X \geq j \iff \operatorname{rank}(J) \leq N - j \iff$ all $(N-j+1) \times (N-j+1)$ minors of $J = 0$. This is a closed condition.

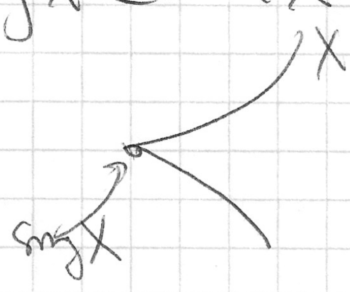
Cor $\operatorname{Sing} X$ is closed in X, the set of smooth points is open in X.

Fact (w/o proof) Any X has at least one smooth point.

Cor The set of smooth points is non-empty, open $\Rightarrow$ dense in X.

Cor $\operatorname{Sing} X =$ proper closed subset of X

$\dim \operatorname{Sing} X < \dim X$.



Ex $X = \{f = 0\} \subset \mathbb{A}^N$ $f(x_1, \ldots x_N) =$ irreducible polynomial

$\dim X = N-1$

$p \in X$ is smooth $\Leftrightarrow \dim T_pX = N-1 \Leftrightarrow$

$\operatorname{rank}\left(\frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_N}\right) = 1 \Leftrightarrow$

at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$

$\operatorname{Sing} X = \left\{\frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_N} = 0, f = 0\right\}$.

Thm Suppose $X = \{f_1 = \ldots = f_k = 0\} \subset \mathbb{A}^N$, $k \leq N$ and at all points $p \in T_pX$ ~~we have~~ the Jacobian matrix J has maximal rank $\operatorname{rank}(J) = k$.

Then X is smooth of dimension $\dim X = N-k$.

Pf If $\operatorname{rank}(J) = k$ then $\dim T_p X = N - \operatorname{rank}(J) = N-k$

Therefore $\dim X \leq \dim T_pX = N-k$.

On the other hand, all components of X have dimension at least $N-k$, so $\dim X \geq N-k$.

Therefore $\dim X = N-k$ and X is smooth.

Thm Suppose $X = \{f_1 = \ldots = f_k = 0\} \subset \mathbb{A}^N$.

$X = \bigcup_i X_i$ irreducible components

Suppose at one point $p \in X_i$ we have $\operatorname{rank}(J_p) = k$.

Then $\dim X_i = N-k$.

__Proof:__ We have $\dim X_i \geq N-k$ since X is cut out by k equations. On the other hand, $\dim T_p X_i \leq \dim T_p X = N-k$ so $\dim X_i \leq N-k$. Therefore $\dim X_i = N-k$.

__Ex__ $\{x_1^2 + \dots + x_n^2 = 1\} = X$

$J = (2x_1 \ \dots \ 2x_n)$ rank $J = 1$ unless $(x_1, \dots, x_n) = 0$

But $(x_1, \dots, x_n) = 0$ is not in $X \Rightarrow X$ smooth, $\dim = n-1$.

$\{x_1^2 + \dots + x_n^2 = 0\} = Y$ has one singular pt at 0.

__Ex__ $X = \{x_1^2 + \dots + x_r^2 = 0\}$ in $\mathbb{A}^n$, $r < n$

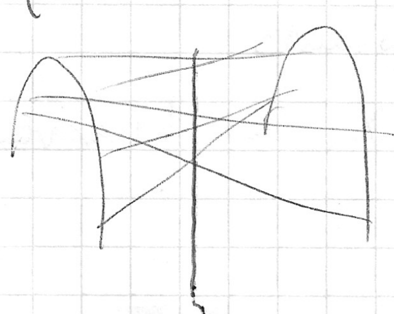
$J = (2x_1 \dots 2x_r \ 0 \dots 0)$

$\operatorname{Sing} X = (0 \dots 0, x_{r+1} \dots x_n)$ ~~[illegible]~~, all such ~~[illegible]~~ points are in X.

__Ex__ Whitney umbrella $\{x^2 + y^2 z = 0\} = X$

$J = (2x \ \ 2yz \ \ y^2)$

$\operatorname{Sing} X = \{x = 0, 2yz = 0, y^2 = 0\} = \{x = y = 0\}$ line



Ex Quadrics in $\mathbb{A}^n$ $\quad$ $\mathbb{K}$ = alg. closed, char $\neq 2$

$$F = \sum Q_{ij} x_i x_j + \sum b_i x_i + c = 0 \qquad Q_{ij} = Q_{ji}$$

$$\frac{\partial F}{\partial x_i} = 2 \sum Q_{ij} x_j + b_i \qquad \left(\begin{array}{l} i=j \text{ gives } Q_{ii} x_i^2 \to 2 Q_{ii} x_i \\ i \neq j \text{ gives } 2Q_{ij} x_i x_j \to 2 Q_{ij} x_j \end{array} \right)$$

Critical point $\Leftrightarrow 2Q \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = - \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

rank $Q = n \Rightarrow$ one point $-\frac{1}{2} Q^{-1} \vec{b}$, may or may not be in $\{F=0\}$ depending on c.

rank $Q = r < n \Rightarrow$ affine hyperplane of dim $= n-r$.

Ex Quadrics in $\mathbb{P}^n$

$$F = \sum_{i,j=0}^{n} Q_{ij} x_i x_j \qquad [x_0 : \ldots : x_n] = \text{homogeneous coord.}$$

In chart $\{x_0 \neq 0\}$ WLOG $x_0 = 1$

$$\sum_{i,j=0}^{n} Q_{ij} x_i x_j = \sum_{i,j=1}^{n} Q_{ij} x_i x_j + 2 \sum_{i=1}^{n} Q_{0i} x_i + Q_{00}$$

Critical points
$$\begin{cases} 2 Q_{ij} x_j + 2 Q_{0i} = 0 & \left(\frac{\partial F}{\partial x_i}\right) \\ \underbrace{\sum Q_{0i} x_i}_{\text{substitute in } F} + Q_{00} = 0 & \leftarrow \text{point in } X \end{cases}$$

$$\Leftrightarrow \quad Q = \left(\begin{array}{c|c} Q_{00} & Q_{0i} \\ \hline Q_{i0} & Q_{ij} \end{array} \right) \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

So F is smooth $\Leftrightarrow \det Q \neq 0$.