Lecture 23 | A *hyperplane* in $\mathbb{P}^n$ is given by one linear equation $\{a_0x_0+\dots+a_nx_n=0\}$

The set of *all* hyperplanes is the *dual* projective space $(\mathbb{P}^n)^\vee$ with coords $[a_0:\dots:a_n]$

Thm (Bertini) Suppose $X\subset\mathbb{P}^n$ is irreducible and smooth. Then there is an open dense subset $U\subset(\mathbb{P}^n)^\vee$ such that for all $H\in U$ the intersection $U\cap X$ is *smooth*.

Rmk For H "in general position" the intersection $X\cap H$ is smooth.

Proof: (1) Pick $p\in X$, assume $H\ni p$. WLOG $p\in\{x_0\neq0\}\subset\mathbb{P}^n$

$\dim X=d$

We have the following cases:

* $X\subset H$, then $T_pX\subset T_pH=H$
* all components of $X\cap H$ have $\dim=d-1$

p is smooth in $X\cap H \iff \dim T_p(X\cap H)=d-1$.

$X=\{f_1=\dots=f_k=0\}$ $\quad X\cap H=\{f_1=\dots f_k=a_0\cancel{x_0}+\dots+a_{n-1}x_{n-1}+a_nx_n=0\}$

$$J=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_n}\end{pmatrix}$$

$\operatorname{rank} J(p)=n-d$

$X\cap H$ smooth at $p \iff$ $\operatorname{rank}\tilde{J}=n-d+1$.

$$\tilde{J}=\begin{array}{c}\begin{array}{ccc}a_1 & \dots & a_n\end{array}\\ \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n}\\ & \dots & \\ \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_n}\end{pmatrix}\end{array}$$

$X \cap H$ singular at $p \iff \operatorname{rank} \tilde{J}(p) = n-d$

$\iff (a_1, \dots, a_n) \in \operatorname{rowspan}(J) \iff \boxed{H \supset T_p X}$

(2) Consider the space of pairs

$Z = \{(p,H) : p \in X,\ X \subset H$ or $X \cap H$ singular at $p\}$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & (\mathbb{P}^n)^\vee \end{tikzcd}$$

The fiber $\pi_1^{-1}(p)$ is the set of hyperplanes H containing $T_p X = \mathbb{P}\left(T_p\mathbb{P}^n / T_p X\right)^\vee \simeq \mathbb{P}^{n-d-1}$.

In fact, π_1 is a locally trivial fibration, so

$$\dim Z = \dim X + (n-d-1) = d + (n-d-1) = n-1.$$

Now define $U = (\mathbb{P}^n)^\vee \setminus \overline{\pi_2(Z)}$

$\dim \overline{\pi_2(Z)} \le \dim Z = n-1$ so $\overline{\pi_2(Z)}$ = proper closed subset of $(\mathbb{P}^n)^\vee$

$\Rightarrow U$ is nonempty open $\Rightarrow$ dense. $\square$

Differentials | A = algebra over k

Ω_A = A module generated by symbols df, $f \in A$

such that
- $d(f_1 + f_2) = df_1 + df_2$
- $d(c) = 0$ for $c \in k$
- $d(fg) = f\,d(g) + g\,d(f)$

Ex $A = \mathbb{K}[x_1 \ldots x_n]$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i \quad \text{(by product rule + induction)}$$

$\Rightarrow \Omega_A = A\langle dx_1 \ldots dx_n\rangle$

Ex $X = \{ f_1 = \ldots = f_k = 0 \} \subset \mathbb{A}^n$

$$A = A(X) = \frac{\mathbb{K}[x_1 \ldots x_n]}{(f_1 \ldots f_k)}$$

Thm $$\Omega_X = \Omega_{A(X)} = \frac{\mathbb{K}[x_1 \ldots x_n]\langle dx_1 \ldots dx_n\rangle}{(f_1 \ldots f_k, df_1, \ldots df_k)}$$

Pf Again by product rule, Ω_X is generated by dx_i over $A(X)$. Since $f_i = 0$ in $A(X)$ and $d(0) = 0$, we get $df_i = 0$ in Ω_X.

More generally, assume $\sum f_i g_i \in I(X)$, then

$$d(\sum f_i g_i) = \sum \underbrace{f_i}_{0} d(g_i) + \sum g_i \underbrace{df_i}_{0 \text{ in } \Omega_X} = 0$$

So these are all the relations.

Ex $\{x^2 = y^3\}$

$$\Omega_X = \frac{A(X)\langle dx, dy\rangle}{\langle 2x\,dx = 3y^2 dy\rangle}$$

If $x \neq 0$ then $dx = \frac{3y^2}{2x} dy$

If $y \neq 0$ then $dy = \frac{2x}{3y^2} dx$.

__Universal property__ A __derivation__ $\partial : A \to M$ valued in an A-module M is a $\mathbb{K}$-linear map such that

$$\partial(f_1 f_2) = \partial(f_1) f_2 + f_1 \partial(f_2).$$

__Fact/exercise__ Any derivation $\partial : A \to M$ factors through Ω_A:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\partial"'] & \Omega_A \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

$\varphi(f\,dg) = f\,\partial(g)$

$\varphi = A$-linear map.

__Lemma__ m_p = maximal ideal at p

Then $\Omega_X / m_p \Omega_X \simeq m_p / m_p^2 \simeq (T_p X)^*$

from last time.

__Proof__ WLOG $p = (0, \ldots, 0)$ $\quad A(X) = \mathbb{K}(x_1 \ldots x_n) / (f_1 \ldots f_k)$

$$f_i = \underbrace{\sum a_{ij} x_j}_{\text{linear part.}} + \ldots \qquad a_{ij} \in \mathbb{K}$$

$$m_p / m_p^2 = \frac{\mathrm{Span}(x_1 \ldots x_n)}{(\sum a_{ij} x_j)}$$

$$df_i = \sum a_{ij}\, dx_j + \underbrace{\cancel{\cdots} + \ldots}_{m_p \Omega_X}$$

$\varphi \mapsto d\varphi$ canonical iso

$$\frac{\Omega_X}{m_p \Omega_X} = \frac{\mathrm{Span}(dx_1 \ldots dx_n)}{(\sum a_{ij}\, dx_j)}$$

So $\Omega_X / m_p \Omega_X \simeq (T_p X)^*$ $\quad \square$.