Lecture 24

Ω_X = differentials on X

Ex $X = \mathbb{P}^1$ with coord. $[x_0 : x_1]$ $\quad \omega \in \Omega_{\mathbb{P}^1}$

Two charts $\quad \{x_0 \neq 0\}$ with $z = \frac{x_1}{x_0}$ $\rightsquigarrow \omega = f(z)\,dz$

$\{x_1 \neq 0\}$ with $w = \frac{x_0}{x_1}$ $\rightsquigarrow \omega = g(w)\,dw$

$$w = \frac{1}{z} \Rightarrow f(z)\,dz = g\left(\tfrac{1}{z}\right) d\left(\tfrac{1}{z}\right) = g\left(\tfrac{1}{z}\right)\left(-\frac{dz}{z^2}\right)$$

$$\underbrace{z^2 f(z)}_{\text{poly in } z} = \underbrace{g\left(\tfrac{1}{z}\right)}_{\text{poly in } \frac{1}{z}}$$

This is not possible unless $\omega = 0$.

So there are **no** global differentials ($=$1-forms) on $\mathbb{P}^1$.

Ex $X = \{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - 2x_2)\} \subset \mathbb{P}^2$ cubic curve

In chart $\{x_2 \neq 0\}$ $\quad x = \frac{x_0}{x_2}$ $\quad y = \frac{x_1}{x_2}$

So $\{y^2 = \underbrace{x(x-1)(x-2)}_{p(x)}\} \subset \mathbb{A}^2$

Claim X is smooth.

Pf In chart $\{x_2 \neq 0\}$ $\quad J = \begin{pmatrix} -p'(x) & 2y \end{pmatrix}$

If $y \neq 0$ then $\operatorname{rank}(J) = 1$

If $y = 0$ then $p(x) = 0 \Rightarrow p'(x) \neq 0 \Rightarrow \operatorname{rank}(J) = 1$

So $X \cap \{x_2 \neq 0\}$ is smooth.

If $X_2=0$ we get $X_0^3=0 \Rightarrow [0:1:0]$ unique point at ∞.

In chart $\{X_1\neq 0\}$ $X_2 = X_0(X_0-X_2)(X_0-2X_2)$

Linearization at $(X_0=X_2=0)$: $X_2=0 \Rightarrow$ smooth at $[0:1:0]$.

Now we study differentials on X

In $\{X_2\neq 0\}$ we get $y^2=p(x)$

$$\boxed{2y\,dy = p'(x)\,dx}$$

<u>Claim</u> $\frac{dx}{y}$ defines a global 1-form on ~~$X \cap \{X_2\neq 0\}$~~.

* $y\neq 0 \Rightarrow \frac{dx}{y}$ defined
* $y=0 \Rightarrow p(x)=0 \Rightarrow p'(x)\neq 0$ $\quad \frac{dx}{y} = \frac{2\,dy}{p'(x)}$

<u>At ∞</u> $\quad X_1^2X_2 = X_0(X_0-X_2)(X_0-2X_2)$

$$\frac{dx}{y} = \frac{d(X_0/X_2)}{(X_1/X_2)} = \frac{X_2\,dX_0 - X_0\,dX_2}{X_2^2\cdot(X_1/X_2)} = \boxed{\frac{X_2\,dX_0 - X_0\,dX_2}{X_1X_2}}$$

$X_1=1$ $\quad X_2 = X_0(X_0-X_2)(X_0-2X_2) = X_0^3-3X_0^2X_2+2X_0X_2^2$

$$dX_2 = (3X_0^2-6X_0X_2+2X_2^2)\,dX_0 + (-3X_0^2+4X_0X_2)\,dX_2$$

$$dx_2 = \frac{(3x_0^2 - 6x_0x_2 + 2x_2^2)\,dx_0}{(1 + 3x_0^2 - 4x_0x_2)}$$

$$\omega = \frac{x_2 dx_0 - x_0 dx_2}{x_2} = dx_0 - \frac{x_0 dx_2}{x_2} = dx_0 - \frac{(3x_0^3 - 6x_0^2x_2 + 2x_0x_2^2)\,dx_0}{x_2(1 + 3x_0^2 - 4x_0x_2)}$$

From the equation $x_0^3 = x_2 + 3x_0^2x_2 - 2x_0x_2^2 \Rightarrow x_0^3$ is divisible by $x_2 \Rightarrow \omega$ is regular as long as $1 + 3x_0^2 - 4x_0x_2 \neq 0$. This is an open subset containing $[0 : 1 : 0]$ so we are done.

<u>Claim</u> Any global 1-form on X is proportional to $\frac{dx}{y}$

$$\Gamma(\Omega^1_X) = \langle \tfrac{dx}{y} \rangle \quad \dim = 1$$

<u>Proof</u> ~~At~~ At every point $\dim \Omega^1_X(p) = \dim (T_pX)^* = 1$

Given some form ω', consider $\frac{\omega'}{\omega}$. Since $\omega \neq 0$ this is a well defined function on X. But we proved that any global function on a projective variety is a constant! So $\omega' = c \cdot \omega$, $c \in K$.

<u>Def</u> Assume X = smooth projective curve ($\dim = 1$)

The <u>geometric genus</u> $p_g(X)$ is defined as

$$p_g(X) = \dim \left\{ \begin{array}{c} \text{global 1-forms} \\ \text{on } X \end{array} \right\} = \dim \Gamma(\Omega^1_X)$$

Ex $p_g(\mathbb{P}^1) = 0$

$$p_g(\{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - 2x_2)\}) = 1.$$

Cool fact

$X_{\mathbb{C}}$ = smooth complex curve over $\mathbb{C}$

$\Rightarrow$ smooth orientable surface in $\mathbb{CP}^2$

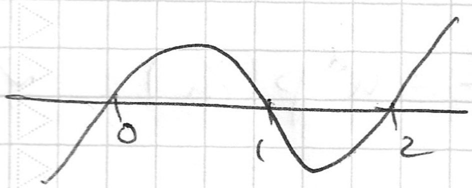
$\mathbb{CP}^2$ is compact $\Rightarrow$ $X_{\mathbb{C}}$ compact.

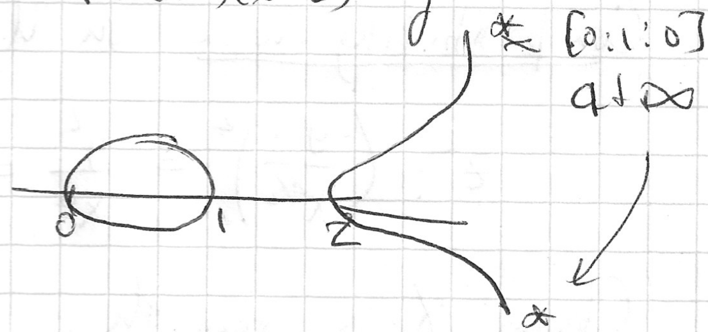
$\Rightarrow$ (classification of surfaces)

genus g surface.

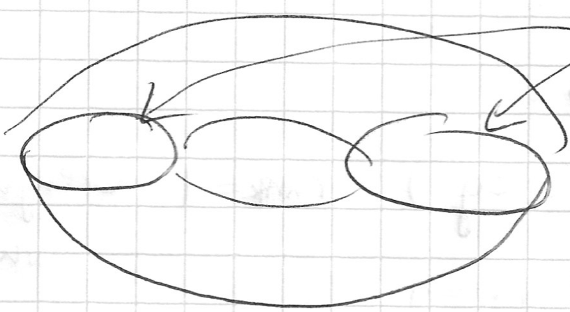
Fact $p_g(X)$ = genus of $X_{\mathbb{C}}$.

Ex $y^2 = x(x-1)(x-2)$ $\qquad \pm\sqrt{x(x-1)(x-2)} = y$





$\Rightarrow$



$X_{\mathbb{R}} \subset X_{\mathbb{C}} = T^2$.