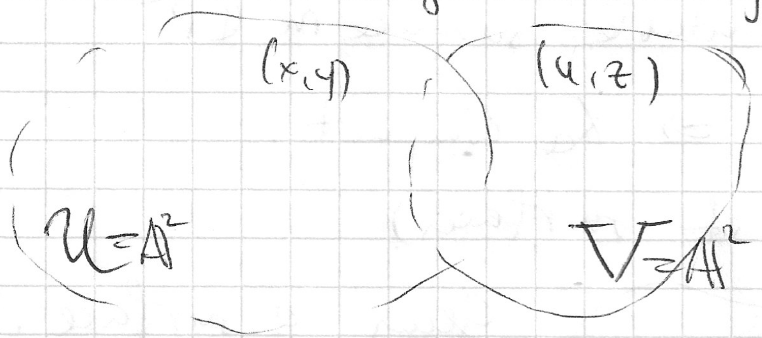
Lecture 25 | Hyperelliptic curves

Choose $p(x)$ = polynomial of degree d with distinct roots.

Today: assume d is even, odd case similar

Consider (abstract) algebraic variety X covered by two charts:



$X \subset (\mathcal{U} \cup \mathcal{V})$

Transition functions: $u = \frac{1}{x}$, $z = \frac{y}{x^{d/2}}$

Equations: $y^2 = p(x)$ in $\mathcal{U}$

$z^2 = u^d p(1/u)$ in $\mathcal{V}$

Compatibility check: on $\mathcal{U} \cap \mathcal{V}$ we have

$$z^2 = \left(\frac{y}{x^{d/2}}\right)^2 = \frac{y^2}{x^d} = \frac{p(x)}{x^d} = u^d p(1/u) \quad \checkmark$$

Claim X is smooth.

Proof Check in charts:

$\mathcal{U}$: $J = \begin{pmatrix} -p'(x) & 2y \end{pmatrix}$ rank $=1$ everywhere on $X \cap \mathcal{U}$

If $y \neq 0$ then $rk = 1$

If $y = 0$ then $p(x) = 0 \Rightarrow p'(x) \neq 0 \Rightarrow$ rank $=1$

simple roots

V: same! Note $u^d p(1/u)$ is also a degree d polynomial with distinct roots

Geometric genus

$$y^2 = p(x) \qquad 2y\,dy = p'(x)\,dx$$

Define $\omega = \dfrac{dx}{y} = \dfrac{2\,dy}{p'(x)}$

regular when $y \neq 0$

regular when $p'(x) \neq 0$

So ω is regular everywhere in $\mathcal{U}$.

Change coordinates to the other chart:

$$\omega = \frac{dx}{y} = \frac{d(1/u)}{(z/u^{d/2})} = -\frac{du}{u^2}\cdot\frac{u^{d/2}}{z} = -\frac{u^{\frac{d}{2}-2}\,du}{z}$$

regular when $z \neq 0$.

Note: "points at ∞" $\stackrel{\text{def}}{=}$ points in $X \cap V$ which are not in $\mathcal{U}$

have $u = 0$, $z^2 = u^d\left(\frac{1}{u^d} + \dots\right) = 1 + \dots$

so $u = 0$, $z = \pm 1$ two points at ∞.

At both $z \neq 0 \Rightarrow \omega$ is regular everywhere!

Furthermore, $x^k\omega = \dfrac{x^k\,dx}{y} = -\dfrac{u^{\frac{d}{2}-2}\,du}{z \cdot u^k}$

always regular in $\mathcal{U}$

regular at ∞ ($u=0$) if $k \leq \frac{d}{2} - 2$.

$y\omega = dx = -\dfrac{du}{u^2}$ = not regular.

Conclusion: $\Omega_X = \mathrm{Span}(\omega, x\omega, \ldots, x^{\frac{d}{2}-2}\omega)$

$$\boxed{p_g(X) = \dim \Omega_X = \frac{d}{2} - 1 = \frac{d-2}{2}.}$$

Exercise Find a similar construction for odd d. ($d=3$ has to be true)

Why hyperelliptic curves are special?

(1) We have an involution $\sigma : X \longrightarrow X$

$\sigma^2 = id$

$\sigma(x,y) = (x,-y)$ $\quad y^2 = p(x)$ preserved ✓

$\sigma(u,z) = (u,-z)$ $\quad z^2 = u^d p(1/u)$ preserved ✓

We have algebraic $\quad u = 1/x$, $z = \frac{y}{x^{d/2}}$ ok.

$\mathbb{Z}_2$ action on X

(2) We have a map $\pi : X \longrightarrow \mathbb{P}^1$

$\pi(x,y) = x$

$u = 1/x$ two charts on $\mathbb{P}^1$

$\pi(u,z) = u$

(3) π^{-1}(point on $\mathbb{P}^1$) = (one $\mathbb{Z}_2$ orbit on X) = $\begin{cases} \text{one point } (x,0) \text{ if } x = \text{root of } p(x) \\ 2 \text{ points otherwise.} \end{cases}$

This means $X/\mathbb{Z}_2 \simeq \mathbb{P}^1$

(incl 2 pts at ∞)

Also, topologically we have a 2:1 cover

$$(X - \text{roots of } p) \xrightarrow[\pi]{2:1} (\mathbb{P}^1 - \text{roots of } p)$$

So π is an example of a "branched cover".

<u>Fact</u> Any smooth curve X satisfying ①–③ is isomorphic to a hyperelliptic curve.

Recall that $X_{\mathbb{C}}$ = smooth $\mathbb{C}$-curve = real surface of some genus g

How to compute g topologically? Use Euler characteristic χ.

<u>Recall</u>
* C = CW complex $\chi(C) = \sum (-1)^i \#(i\text{-cells})$ does not depend on cell decomposition
* $\chi(\text{genus } g \text{ surface}) = 2-2g$ $\chi(\mathbb{CP}^1) = \chi(S^2) = 2$
* If $Z \subset C$ closed then $\chi(C) = \chi(Z) + \chi(C \setminus Z)$.

Now $\chi(\mathbb{P}^1 - \text{roots of } p) = 2-d$

$\chi(X - \text{roots of } p) = 2 \cdot \chi(\mathbb{P}^1 - \text{roots of } p) = 2(2-d)$

$\chi(X) = \chi(X - \text{roots of } p) + d = 4 - 2d + d = 4 - d$

Now $4 - d = 2 - 2g$

$2g = d - 2$

$g = \frac{d-2}{2}$ Same as p_g from algebra!

How to prove $p_g = g$ for general curve X?

<u>Idea:</u> find a branched cover $X \to \mathbb{P}^1$

prove a formula for p_g and $\chi(X) = 2 - 2g$ in terms of this cover and compare.