Lecture 26 | Divisors and Picard group

Pic X → for X affine, characterizes when $A(X)$ is UFD

→ for any X controls line bundles on X.

Def A (Weil) divisor on X is a linear combination $\sum a_i [D_i]$ D_i = closed, irreducible subsets in X

$\dim D_i = \dim X - 1$.

Ex $X = \mathbb{A}^n$, $f \in \mathbb{K}[x_1, \dots, x_n]$

Since $\mathbb{K}[x_1, \dots, x_n]$ = UFD, we can write $f = f_1^{m_1} \cdots f_s^{m_s}$

f_i = irreducible distinct

Define $\operatorname{div}(f) = \sum m_i [\{f_i = 0\}]$

irreducible closed $\dim = \dim X - 1$

We want to generalize this example

Thm X = irreducible affine alg. set. The following are equivalent:

(a) $A(X)$ is a UFD

(b) All closed, irreducible subsets of X of $\dim = \dim X - 1$ are hypersurfaces $\{f = 0\}$.

Proof: (a) ⇒ (b) $A(X)$ is a UFD, $Y \subset X$ irred $\dim Y = \dim X - 1$

$I(Y)$ = prime ideal in $A(X)$

Pick $f \in I(Y)$, factor $f = \prod f_i^{m_i}$, f_i = irred.

$X \supset \{f_i = 0\} \supset Y$

$I(Y)$ prime $\Rightarrow f_i \in I(Y)$ for ~~all~~ some i.

$\dim \{f_i = 0\} = \dim X - 1$

If $Y \neq \{f_i = 0\}$ then $\dim Y < \dim \{f_i = 0\} < \dim X - 1$ contradiction

$\Rightarrow Y = \{f_i = 0\}$.

(b) $\Rightarrow$ (a) Recall that f is called irreducible if $xy = f \Rightarrow x$ or y is a unit. We need to prove any f is a <u>unique</u> product of irreducibles.

* Assume f is not irreducible $\Rightarrow f = xy$ and continue. By <u>Noetherian</u> property this process stops and f = product of irreducibles.

* Assume f is irreducible, let us prove (f) is prime.

$\{f = 0\} = \underbrace{\bigcup Y_i}_{\text{irred comp.}}$ $\quad \dim Y_i = \dim X - 1$.

$\Rightarrow$ by (b) we have $I(Y_i) = (g_i)$.

$f \in I(Y_1) \Rightarrow f \in (g_1) \Rightarrow f = g_1 \cdot k_1$, since f is irreducible and g_1 is not a unit $\Rightarrow$ (k is a unit) $(f) = (g_1)$ is prime.

* Now it is easy. Assume

$f_1^{n_1} \cdots f_s^{n_s} = \bar{f}_1^{n_1} \cdots \bar{f}_t^{n_t}$ $\quad f_i, \bar{f}$ irred.

(f_1) is prime, RHS $\in (f_1) \Rightarrow$ one of $\bar{f}_i \in (f_1) \Rightarrow$
$\Rightarrow \bar{f}_i = f_1 \cdot$ unit. Cancel and continue.

This proves uniqueness. $\square$.

Suppose X is smooth* and irreducible

*In fact normal

Fact Suppose X smooth & affine, $Y \subset X$ irreducible

$\dim Y = \dim X - 1$; $f \in A(X)$, $f \neq 0$.

Then one can define the order of vanishing (= divisorial valuation)

$\nu_Y(f)$ with the following properties:

① $\nu_Y(f) \geq 0$

② If f vanishes on Y $\Leftrightarrow f \in I(Y)$ then $\nu_Y(f) > 0$

Otherwise $\nu_Y(f) = 0$.

③ $\nu_Y(fg) = \nu_Y(f) + \nu_Y(g)$

④ If $(f) = I(Y)$ then $\nu_Y(f) = 1$

⑤ If $\nu_Y(f) \geq \nu_Y(g)$ then $\frac{f}{g}$ = regular function on open subset of Y.

$\Leftrightarrow \frac{f}{g} = \frac{f'}{g'}$, $g' \notin I(Y)$.

Ex $Y = \{y = 0\} \subset \mathbb{A}^2_{x,y}$

$f = a_0(x) + y a_1(x) + y^2 a_2(x) \dots$

$\nu_Y(f) = \min\{i : a_i(x) \neq 0\}$

$\nu_Y(xy^2) = 2$ $\quad \nu_Y(x+y) = 0$.

Ex $f \in \mathbb{K}[x_1 \dots x_n]$ $\quad Y \subset \mathbb{A}^n = X$ irred $\dim Y = n-1$.

$f = \prod f_i^{m_i}$ (irred) $\quad \nu_Y(f) = \begin{cases} m_i, & Y = \{f_i = 0\} \\ 0, & \text{otherwise.} \end{cases}$

Def $f \neq 0$, $f \in A(X)$ then $\boxed{\operatorname{div}(f) = \sum_Y \nu_Y(f)[Y]}$

Note Only finitely many Y contribute,

Y = component of $\{f = 0\}$,

Def $\frac{f}{g}$ = rational function $\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g)$.

Lemma (a) For all f, g $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$

(b) $\operatorname{div}\left(\frac{f}{g}\right)$ is a well defined homomorphism

$$[\operatorname{Frac} A(X)^*, \cdot] \longrightarrow (\operatorname{Div}(X), +).$$

Proof (a) $\operatorname{div}(fg) = \sum \nu_Y(fg)[Y] = \sum_Y (\nu_Y(f) + \nu_Y(g))[Y]$

$= \sum_Y \nu_Y(f)[Y] + \sum_Y \nu_Y(g)[Y] = \operatorname{div}(f) + \operatorname{div}(g)$

(b) $\frac{f}{g} \sim \frac{fh}{gh}$ $\operatorname{div}(fh) - \operatorname{div}(gh) = \operatorname{div}(f) + \operatorname{div}(h) - \operatorname{div}(g) - \operatorname{div}(h)$

$= \operatorname{div}(f) - \operatorname{div}(g) = \operatorname{div}\left(\frac{f}{g}\right)$.

Similarly $\operatorname{div}\left(\frac{f}{g} \cdot \frac{f'}{g'}\right) = \operatorname{div}\left(\frac{f}{g}\right) + \operatorname{div}\left(\frac{f'}{g'}\right)$.

Def A principal divisor is $\operatorname{div}\left(\frac{f}{g}\right)$, $\frac{f}{g}$ = rational fn on X.

This is a subgroup $\operatorname{Prin}(X) \subset \operatorname{Div}(X)$

Def Divisor class group

$$\operatorname{Cl}(X) = \frac{\operatorname{Div}(X)}{\operatorname{Prin}(X)}$$

all divisors

principal divisors.