Lecture 27 | Recap

X = irreducible

A **divisor** is a linear combination $D = \sum a_i [Y_i]$

Y_i = irreducible, codim $= 1$, $a_i \in \mathbb{Z}$

$\operatorname{div}(f) = \sum \nu_Y(f)[Y]$ $\nu_Y(f)$ = order of vanishing of f on Y

$\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g) \leftarrow$ Principal divisors

Def Divisor class group $Cl(X) = \dfrac{\operatorname{Div}(X)}{\operatorname{Prin}(X)}$.

Ex $Cl(\mathbb{A}^1) = 0$ $\operatorname{Div}(X) = \{\sum a_i [p_i]\} \leftarrow$ points

$\frac{f}{g} = \prod (x - p_i)^{a_i}$ $\operatorname{div}\left(\frac{f}{g}\right) = \sum a_i [p_i]$

So $\operatorname{Div}(X) = \operatorname{Prin}(X)$.

Thm X = affine, smooth. Then $A(X)$ is a UFD $\Leftrightarrow Cl(X) = 0$

Pf last time we proved:

$A(X)$ is a UFD $\Leftrightarrow$ any $Y \subset X$ closed, irred, codim 1
$(*)$ satisfies $I(Y) = (f)$.

① Assume $A(X)$ = UFD

$$D = \sum a_i [Y_i] = \sum a_i [\{f_i = 0\}] = \sum a_i \operatorname{div}(f_i) =$$
$$= \operatorname{div}\left(\prod f_i^{a_i}\right) \Rightarrow Cl = 0.$$

② Assume $Cl(X) = 0$, so any divisor is principal.

$Y \subset X$ closed, irred, codim 1, $[Y] = \operatorname{div}\left(\frac{f}{g}\right)$ for some $f, g \in A(X)$

$$\nu_Y(f) - \nu_Y(g) = 1 \Rightarrow \frac{f}{g} = \frac{f'}{g'}, \quad g' \notin I(Y)$$

Define $U_0 = \{g' \neq 0\} \cap Y$ $\quad Z_0 = Y \cap \{g' = 0\}$.

$\dim Z_0 = \dim Y - 1 = \dim X - 2$

or $Z_0 = \emptyset$

Define $\{g = 0\} = \bigcup Y_i$ = component.

For $Y_i \neq Y$, $\nu_{Y_i}(f) = \nu_{Y_i}(g) \Rightarrow \frac{f}{g} = \frac{f^{(i)}}{g^{(i)}}$ $g^{(i)} \notin I(Y_i)$

so $Y_i \cap \{g^{(i)} = 0\}$ have $\dim = \dim X - 2$.

So $\frac{f}{g}$ is regular outside some closed subset of codim 2

Since ~~X~~ is smooth, $\frac{f}{g}$ is regular everywhere.

So $Y = \{\varphi = 0\}$ for a regular fn $\varphi = \frac{f}{g}$. $\square$

Thm $Cl(\mathbb{P}^n) = \mathbb{Z}$

Proof $Y \subset \mathbb{P}^n$ irred $\dim Y = n-1$ $\rightsquigarrow \tilde{Y} \subset \mathbb{A}^{n+1}$ irred, $\dim \tilde{Y} = n$.

$\tilde{Y} = \{g = 0\}$ for some homogeneous irred. polynomial g

Define $\deg(Y) = \deg(g)$.

$$\deg : \mathrm{Div}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

$$\prod g_i^{a_i} \Longleftrightarrow \sum a_i [Y_i] \longrightarrow \sum a_i \deg(Y_i)$$

g_i = eq. of Y_i

$\mathrm{Prin}(\mathbb{P}^n)$ = rational fn of deg 0 $= \ker(\deg)$

By Isomorphism theorem, $Cl(\mathbb{P}^n) = \frac{\mathrm{Div}(\mathbb{P}^n)}{\mathrm{Prin}(\mathbb{P}^n)} = \mathbb{Z}$.

Ex $X = \{y^2z = x(x-z)(x-2z)\} \subset \mathbb{P}^2$

Smooth cubic curve, one point at infinity $[0:1:0] = p_0$.

$Cl(X) = ?$

Step 1: $\deg : Div(X) \to \mathbb{Z}$

$$\deg(\sum a_i [p_i]) = \sum a_i$$

Fact If $\frac{f}{g}$ rational fn (degree 0) then $\deg(\mathrm{div}(\frac{f}{g})) = 0$.

Total # zeros = # poles.

Cor $Prin(X) \subset Ker(\deg)$

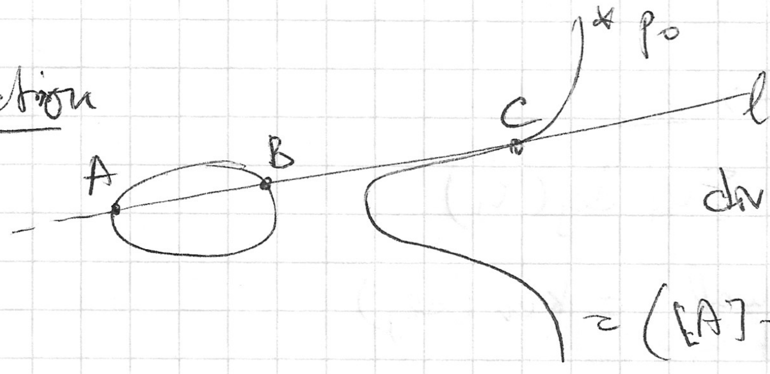
Define $Cl^0 = \frac{Ker(\deg)}{Prin(X)}$. We have $Cl^0 \subset Cl$, $\boxed{Cl/Cl^0 \simeq \mathbb{Z}}$

Step 2. We have a map $X \to Cl^0(X)$

$$p \longmapsto [p] - [p_0]$$

Thm This is an isomorphism, $X \simeq Cl^0(X)$. In particular, X has a structure of abelian group!

Construction

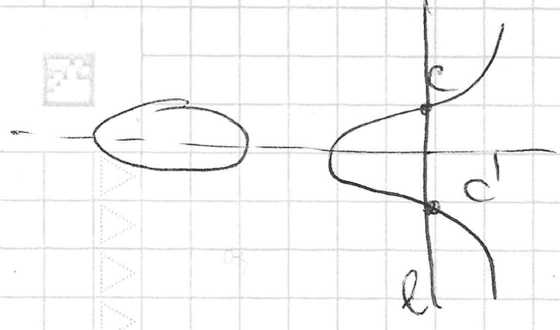


$\ell = \{ax + by + cz = 0\}$

$$\mathrm{div}\left(\frac{\ell}{z}\right) = [A] + [B] + [C] - 3[p_0]$$

$$= ([A] - [p_0]) + ([B] - [p_0]) + ([C] - [p_0]) = 0$$

in Cl^0.



$\mathrm{div}\left(\frac{l}{z}\right) = [C] + [C'] + [p_0] - 3[p_0] =$
$= ([C] - [p_0]) + ([C'] - [p_0])$

Define $A + B = C'$, $C + C' = 0$

Given two points A, B draw the line $l = AB$, it intersects X at 3 points. If $C = 3$rd pt, define $C' =$ reflect C in y-axis.

Fact This is associative! Clearly commutative.
identity $= 0 = p_0$ inverse $-C = C'$.

By the above, the map $X \to Cl^0(X)$ is a group homomorph.
Surjective: given $D = \sum a_i [p_i]$ with $\sum a_i = 0$
we can write $D = \sum a_i ([p_i] - [p_0]) = [p] - [p_0]$
where $p = \sum a_i p_i$ wrt group law on X.
Injective w/o proof.

Cor $Cl^0(X)$ is large!

Rmks ① $X =$ smooth genus g curve
$Cl^0(X) =$ Jacobian of $X = 2g$-dimensional $\mathbb{T}^{2g}$
Algebraic variety, $\dim_{\mathbb{C}} = g$, with structure of abelian group. thing

② $X =$ general smooth variety

$\deg : Cl(X) \to H^2(X; \mathbb{Z})$ Ker $=$ interesting.
$\sum a_i [Y_i] \to \sum a_i PD[Y_i]$