Lecture 1 (1/5) What is this class about?

$\mathbb{K}$ = field $\mathbb{K}[x_1 \ldots x_n]$ = polynomial ring

$\mathbb{A}^n = \mathbb{A}^n(\mathbb{K}) = \{(x_1 \ldots x_n) : x_i \in \mathbb{K}\}$ n-dim space over $\mathbb{K}$

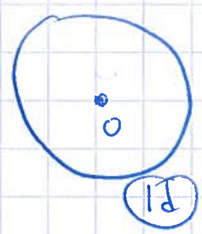
$f_1(x_1 \ldots x_n), \ldots f_m(x_1 \ldots x_n)$ = some polynomials in $\mathbb{K}[x_1 \ldots x_n]$

Def An affine algebraic variety (= alg. set) is defined by a system of polynomial equations

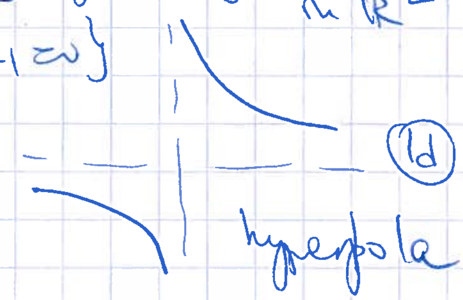
$$Z = \{f_1(x_1 \ldots x_n) = \ldots = f_m(x_1 \ldots x_n) = 0\} \subset \mathbb{A}^n$$

Ex $Z = \{f(x,y) = 0\} \subset \mathbb{A}^2(\mathbb{K})$ $\boxed{\mathbb{K} = \mathbb{R}}$ typically 1d curve in $\mathbb{R}^2$

① $\{x^2 + y^2 - 1 = 0\}$ circle

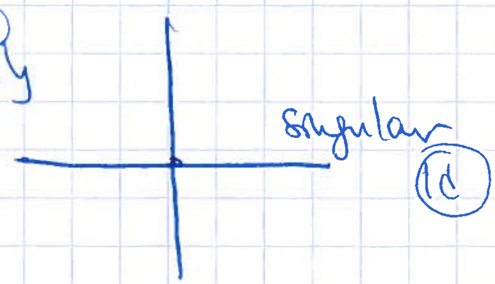


④ $\{xy - 1 = 0\}$



② $\{x^2 + y^2 + 1 = 0\}$ $\emptyset$ empty!

⑤ $\{xy = 0\}$



③ $\{x^2 + y^2 = 0\}$ $\bullet(0,0)$ point 0d

Q: Is it compact? Connected? How many connected components? Smooth? (239) Dim = ?

$\boxed{\mathbb{K}=\mathbb{C}}$ $\Rightarrow$ one complex eq. in $\mathbb{C}^2$, typically 2d surface, smooth, noncompact

$\Rightarrow$ genus g surface with boundary

Q: How to read the genus g from the equation?

NB: $\mathbb{C}$ is algebraically closed

As we will see, this makes lots of things easier and we will often work over alg. closed fields with $\mathbb{C}$ = main example.

$\boxed{\mathbb{K}=\mathbb{Q} \text{ or } \mathbb{K}=\mathbb{F}_p}$ $\Longrightarrow$ rational points / solutions to $f(x,y)=0$ mod p

[Number theory]

Def Projective space

$$\mathbb{P}^n(\mathbb{K}) = \{[z_0 : \dots : z_n]\}/\sim, \text{ not all } z_i = 0$$

$$[z_0 : \dots : z_n] \sim [\lambda z_0 : \dots : \lambda z_n] \quad \lambda \neq 0$$

Fact/exercise $\mathbb{P}^n(\mathbb{K}) = \{\text{all lines in } \mathbb{A}^{n+1}(\mathbb{K}) \text{ through } 0\}$

If $f(z_0, \dots z_n)$ is a homogeneous degree d polynomial then $f(\lambda z_0, \dots \lambda z_n) = \lambda^d f(z_0, \dots z_n)$

So $\{f(z_0, \dots z_n) = 0\}$ is a well defined subset of $\mathbb{P}^n$.

Def A projective algebraic variety is

$Z = \{ f_1(z_0, \dots z_n) = \dots = f_m(z_0, \dots z_n) = 0 \} \subset \mathbb{CP}^n$

where all f_i are homogeneous polynomials.

Rmk Over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ the projective space is compact (in usual topology);

Z = closed subset of $\mathbb{P}^n \Rightarrow$ compact. see HW1 for examples

---

$T \subset \mathbb{K}[x_1, \dots x_n]$ collection of polynomials (not necessarily finite)

Def $Z(T) = \{ (x_1 \dots x_n) \mid f(x_1, \dots x_n) = 0$ for all $f \in T \} \subset \mathbb{A}$

All such Z are called algebraic sets.

Lemma: 1) $T_1 \subset T_2 \Rightarrow Z(T_1) \supset Z(T_2)$ (reverse inclusion)

2) I = ideal generated by $T \Rightarrow Z(I) = Z(T)$

Pf 1) $p \in Z(T_2) \Rightarrow f(p) = 0$ for all $f \in T_2$

$\Rightarrow f(p) = 0$ for all $f \in T_1 \Rightarrow p \in Z(T_1)$

2) $T \subset I \Rightarrow$ by (1) $Z(I) \subset Z(T)$. Let's prove other inclusion.

$f \in I$ $\quad f = \sum f_i g_i$ for $f_i \in T$ $\quad g_i$ any

For $p \in Z(T)$ we have $f_i(p) = 0 \Rightarrow f(p) = 0$.

So $Z(T) \subset Z(I)$

Ex $I = (x,y) \subset \mathbb{K}[x,y]$
$\qquad \| \quad \{xg_1 + yg_2\}$

What is $Z(I)$? If $p \in Z(I)$ ~~then~~ $x(p) = y(p) = 0$
so $p = (0,0)$. On the other hand, all functions in I
vanish at $(0,0) \Rightarrow \boxed{Z(I) = \{(0,0)\}}$

Lemma I_1, I_2 = ideals

(a) $Z(I_1 + I_2) = Z(I_1) \cap Z(I_2)$

(b) $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$

(c) $Z(1) = \emptyset$ $\qquad$ (d) $Z(0) = \mathbb{A}^n$.

Pf (c), (d) clear; (a): $I_1 + I_2$ = ideal gen by $I_1 \cup I_2$.

(b): Assume $p \in Z(I_1) \cup Z(I_2)$, $f \in I_1 \cap I_2$.

Then either $p \in Z(I_1)$ or $p \in Z(I_2) \Rightarrow f(p) = 0$.

So $Z(I_1) \cup Z(I_2) \subset Z(I_1 \cap I_2)$

Conversely, assume $p \in Z(I_1 \cap I_2)$ but $p \notin Z(I_1)$
$p \notin Z(I_2)$

$\exists f \in I_1$ such that $f(p) \neq 0$, $g \in I_2 : g(p) \neq 0$

Then $fg \in I_1 \cap I_2$ since I_1, I_2 are ideals

and $fg(p) \neq 0$, contradiction.
