What is this course about?

Key example: $GL(n,\mathbb{R})$ / $GL(n,\mathbb{C})$ $n\times n$ invertible matrices

(1) This is a group! Lie group

We can study

- subgroups (we will see a lot!)
- conjugacy classes

Ex Conj. classes in $GL(n,\mathbb{C}) \longleftrightarrow$ Jordan normal form

$$B\in GL(n,\mathbb{C}) \xrightarrow[\exists A]{} ABA^{-1} = \begin{pmatrix} \begin{matrix}\lambda_1 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_1\end{matrix} & \\ & \begin{matrix}\lambda_2 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_2\end{matrix} \\ & & \ddots \end{pmatrix}$$

Jordan block

λ_i and the sizes of blocks are determined by B up to permutation of blocks

B invertible $\longrightarrow \lambda_i \neq 0$

λ_i = eigenvalues of B

All blocks have size 1 $\Longrightarrow$ B diagonalizable $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

- Group actions

Ex: $GL(n)$ acts on $\mathbb{R}^n / \mathbb{C}^n$

A = matrix in $GL(n)$ v = vector

$A(v) = A\cdot v$

We can study orbits & stabilizers for this action. (HW #1)

$\{0\}$ is an orbit

$\mathrm{Stab}\{0\} = GL(n)$

$\mathrm{Stab}(v)$ = some subgroup of $GL(n)$

Ex 2 $GL(n)$ acts on itself by conjugation

$A \in GL(n) \quad B \in GL(n)$

$$A(B) = ABA^{-1} \longleftarrow \text{action of } A \text{ on } B$$

$$A_1A_2(B) = A_1A_2B(A_1A_2)^{-1} = A_1A_2BA_2^{-1}A_1^{-1}$$

$$= A_1(A_2BA_2^{-1})A_1^{-1} = A_1(A_2(B))$$

Orbits = conjugacy classes (see above)

$\mathrm{Stab}(B) = \{A : ABA^{-1} = B \iff AB = BA$
$\iff$ all matrices commuting w. $B\}$

Ex 1 = representation of $G(n)$ = linear action

Ex 2 is not ($GL(n)$ is not a vector space).

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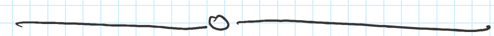
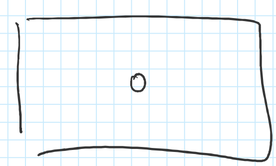
② $GL(n)$ is a topological space! (open subset of $\mathrm{Mat}(n \times n)$)

Ex $GL(1, \mathbb{R}) = \mathbb{R}^* = \{\text{nonzero real numbers}\} = \mathbb{R} \setminus \{0\}$



$GL(1, \mathbb{C}) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$



- Is it compact?

  $GL(n, \mathbb{R})$ not compact

- Is it connected / (path) connected

  $GL(1, \mathbb{R})$ no!

$$\underset{\{\det \neq 0\}}{GL(n, \mathbb{R})} = \underset{\{\det > 0\}}{GL^+(n; \mathbb{R})} \sqcup \underset{\{\det < 0\}}{GL^-(n; \mathbb{R})}$$

$GL(n, \mathbb{R})$ has at least two connected components

$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$

Fact $GL^+(n, \mathbb{R})$ and $GL^-(n, \mathbb{R})$ are connected, so $GL(n, \mathbb{R})$ has exactly two connected components.

(proof later this week).

Fact $GL^+(n,\mathbb{R}) = \{A : \det > 0\}$ is a subgroup of $GL(n,\mathbb{R})$

Proof $\det(A) > 0,\ \det(B) > 0 \Rightarrow \det(AB) > 0$

$$\det(AB) = \det A \cdot \det B$$

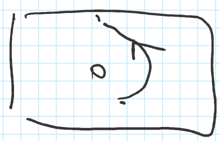
$$\det(A) > 0 \Rightarrow \det(A^{-1}) = \frac{1}{\det A} > 0.$$

This is a special case of a general phenomenon:

G = some Lie group, disconnected

G^+ = connected component containing $\mathbb{1}$, this is always a subgroup.

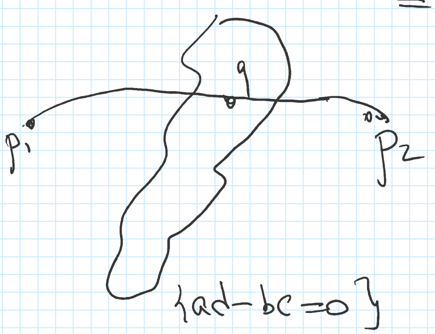
- $GL(1,\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is connected!



$$GL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$$

$$= \mathbb{C}^4 \setminus \{ad - bc = 0\}$$

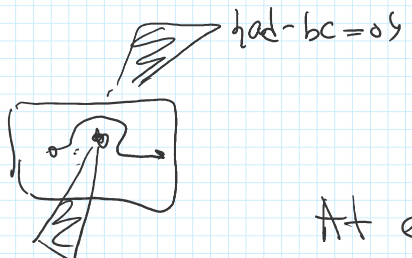
real codim 2 in $\mathbb{C}^4$ = complex codim 1.



Claim: $GL(2,\mathbb{C})$ is connected

Idea of proof: pick two points p_1, p_2 connect them by a path which intersects $\{ad-bc=0\}$ in some number of points. (smooth)

At every pt of intersection we can go around using the transversal 2-plane. ☒



- $\pi_1 = ?$
- Structure maps are continuous (actually, smooth):

$$m : G \times G \longrightarrow G \qquad i : G \longrightarrow G$$
$$(A,B) \longrightarrow AB \qquad A \longrightarrow A^{-1}$$

multiplication   inverse.

multiplication inverse.

③ $\mathfrak{gl}_n = \text{Mat}(n \times n)$ matrices all $n \times n$ (example of a Lie algebra)

- Vector space
- Operation: $[X, Y] = XY - YX$ commutator. bilinear, more interesting properties later.
- GL_n and $\mathfrak{gl}_n$ are closely related — this is one of key ideas!
- GL_n acts on $\mathfrak{gl}_n$ by conjugation:

$A(X) = AXA^{-1}$, this preserves commutators:

$A \in GL_n$, $X \in \text{Mat}$

$$[AXA^{-1}, AYA^{-1}] = AXA^{-1} \cdot AYA^{-1} - AYA^{-1} \cdot AXA^{-1} = A[X, Y]A^{-1}$$

"adjoint representation".

Def A matrix Lie group = closed subgroup of $GL(n, \mathbb{C})$

$G \subset GL(n, \mathbb{C})$
- subgroup
- $A_n \in G$, $\lim_{n \to \infty} A_n = A$ then either $A \in G$ or A not invertible.

Ex $GL(n, \mathbb{R})$ is a matrix Lie group.