Lecture 10 (4/20)

Wednesday, April 20, 2022 1:06 PM

$$U(n) \overset{\curvearrowleft}{=} SU(n) \times U(1)$$

homeomorphic as top. spaces

<u>NOT</u> isomorphic as groups!!

$$SU(n) \longrightarrow U(n) \xrightarrow{\det} U(1)$$

$\cap$ Ker(det)   ↳ group homomorphis

have a section $\sigma : U(1) \longrightarrow U(n)$

$$\alpha \longrightarrow \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

<u>But</u> this section does not commute with $SU(n)$!

<u>So</u>: we have a continuous map

$$SU(n) \times U(1) : (A, \alpha) \longrightarrow A \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

this is <u>a homeomorphism</u>

$$A = B \cdot \begin{pmatrix} \det B^{-1} & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \longleftarrow B$$

$$\alpha = \det B$$

<u>Not a group homomorphism</u> since A does not commute with $\begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

<u>Exercise</u> Prove that $U(n)$ and $SU(n) \times U(1)$ are not isomorphic as groups. <u>Hint</u>: compute $Z(U(n))$ ← center

$$Z(SU(n) \times U(1)) = Z(SU(n)) \times U(1)$$

<u>Recall</u> important Lemma:

$\lambda_i \neq \lambda_j$ $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ commutes with $M \Rightarrow M$ is diagonal.

anything in $Z(U(n))$ or $Z(SU(n))$ is diagonal

Check $Z(U(n)) = \{ \begin{pmatrix} \lambda & \ddots & 0 \end{pmatrix} : |\lambda| = 1 \} = U(1)$

check $Z(U(n)) = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}, |\lambda| = 1 \right\} = U(1)$

$Z(SU(n)) = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}, \begin{matrix} |\lambda| = 1 \\ \lambda^n = 1 \end{matrix} \right\} = \{ n\text{-th roots of } 1 \}$

Examples of Lie algebras

① $G = GL_n$ $\quad \mathrm{Lie}(G) = \mathfrak{gl}_n = \{\text{all } n \times n \text{ matrices}\}$

$\dim \mathfrak{gl}_n(\mathbb{R}) = n^2 \qquad \dim_{\mathbb{R}} \mathfrak{gl}_n(\mathbb{C}) = 2n^2$

② $G = SL_n$ $\quad \mathrm{Lie}(G) = \mathfrak{sl}_n = \{ X : \mathrm{Tr}\, X = 0 \}$

$\dim \mathfrak{sl}_n(\mathbb{R}) = n^2 - 1$

③ $G = O(n)$ $\quad \underset{\mathfrak{o}(n)}{\underset{\shortparallel}{\mathrm{Lie}(G)}} = \{ X : X + X^T = 0 \}$ see last Friday lec.

skew-symmetric matrices

$\dim = \frac{n^2 - n}{2}$

$$\begin{pmatrix} 0 & & x_{ij} \\ & \ddots & \\ -x_{ij} & & 0 \end{pmatrix}$$

$G = SO(n)$ $\quad \mathrm{Lie}(G) = \left\{ X : \begin{matrix} X + X^T = 0 \\ \mathrm{Tr}(X) = 0 \end{matrix} \right\}$ holds automatically since 0's on diagonal.

$\mathfrak{o}(n) = \mathfrak{so}(n).$

<u>Ex</u> $\mathfrak{so}(3) = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$ $\quad \dim = 3$

basis $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$

HW4: compute $[\,,\,]$ in this basis.

④ $G = U(n)$ $\quad \mathrm{Lie}(G) = \mathfrak{u}(n) = \{ X : X^* + X = 0 \}$

④ $G = U(n)$ $\quad \text{Lie}(G) = \mathfrak{u}(n) = \left\{ X : \underset{\overline{X^T}+X=0}{\overset{}{X^* + X = 0}} \right\}$ "Hermitian skew symmetric"

$$\begin{pmatrix} ia_1 & & x_{ij} \\ & ia_2 & \\ -\overline{x_{ij}} & & \ddots \\ & & & ia_n \end{pmatrix}$$

On diagonal, $\bar{x} + x = 0$

$x = ia \quad a \in \mathbb{R}$

Upper-triangular part determines the lower-triangular one.

$$\dim = \underset{\text{diagonal}}{n} + \overset{\mathbb{C}}{2} \cdot \underbrace{\frac{n^2-n}{2}}_{\text{upper-trian}} = n^2$$

$$\text{Lie}(SU(n)) = su(n) = \left\{ X : \begin{matrix} X^* + X = 0 \\ Tr(X) = 0 \end{matrix} \right\}$$

additional condition $a_1 + \dots + a_n = 0$

$\dim su(n) = n^2 - 1$.

⑤ T = (abelian) group of diagonal invertible matrices $= \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$

algebraic torus

$\alpha_i \neq 0$ for all i

Over $\mathbb{R}$: $(\mathbb{R}^*)^n$ $\qquad$ Over $\mathbb{C}$: $(\mathbb{C}^*)^n \supset U(1)^n = (S^1)^n$ n-dim torus.

$\text{Lie}(T) = \{\text{all diagonal matrices}\}$

$\underline{\dim = n}$

Know $A(t) = e^{tX}$ diagonal for all t

$\frac{d}{dt}A(t)$ diagonal $\Rightarrow X$ diagonal.

$X, Y \in \text{Lie}(T) \Rightarrow XY - YX = 0$

abelian Lie algebra $[\ ,\ ] = 0$.

⑥ B = (Borel) subgroup of upper-triangular invertible matrices $\begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$

⑥ $B = $ (Borel) subgroup

upper-triangular invertible matrices $\begin{pmatrix} \ddots & * \\ 0 & \ddots \ \ d_n \end{pmatrix}$

$$\mathrm{Lie}(B) = \begin{pmatrix} x_1 & & * \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \qquad \dim = n + \frac{n^2-n}{2} = \frac{n^2+n}{2}.$$

Thm G = matrix Lie group $\Rightarrow$ G is a smooth manifold.

$$\mathrm{Lie}(G) \simeq T_{\underset{\text{identity matrix}}{I}} G.$$

$$\underset{\text{as a manifold}}{\dim G} = \underset{\text{as a vector space}}{\dim \mathrm{Lie}(G)}$$

Proof: Friday.

How to prove that $O(n) = \{A : A^T A = I\}$ is a smooth mfd?

Recall $\mathbb{R}^N$, equations $F_1(x_1 \dots x_N) = \dots = F_k(x_1 \dots x_N) = 0$

Implicit Function Theorem: $J = \left(\dfrac{\partial F_i}{\partial x_j}\right)$ Jacobi matrix

If $\mathrm{Rank}(J) = k$ for all points of

$M = \{F_1 = \dots = F_k = 0\}$

then M is a smooth manifold of $\dim = N - k$.

① $F(x + \delta x) = F(x) + \sum \dfrac{\partial F}{\partial x_i} \cdot \delta x_i + \dots$ $\Big\{$

Can use it to get all partial derivatives.

Eqn $A^T A = I$, replace A by $A + \delta A$

$$(A + \delta A)^T \cdot (A + \delta A) = A^T A + \underbrace{A^T \cdot \delta A + (\delta A)^T A}_{J \cdot \delta A} + \text{higher order}$$

J = Jacobi matrix.

$\cdot\,\delta A$

$J =$ Jacobi matrix.

$$J\cdot\delta A = 0 \iff A^T\delta A + (\delta A)^T A = 0. \qquad (**)$$

$N = n^2$ ambient space $\qquad k = \#$ equations $= \frac{n^2+n}{2}$

Want to prove $\operatorname{Rank}(J) = \frac{n^2+n}{2}$

$$\iff \dim\{\underset{\parallel}{\delta A} : J\cdot\delta A = 0\} = N - k = \frac{n^2-n}{2}.$$

space of solutions of $(**)$
for fixed $A \in O(n)$.

② Solve $(**)$ at $A = I$ first!

$$\delta A + (\delta A)^T = 0 \quad = \operatorname{Lie}(O(n)),\ \dim = \frac{n^2-n}{2} \text{ correct.}$$

$\Rightarrow \operatorname{Rank}(J)$ at $A = I$ is maximal
by Implicit Function Thm we can choose a neighborhood of $I \simeq$ open subset of $\mathbb{R}^{\frac{n^2-n}{2}}$.

Key idea: To get a neighborhood of A, translate the nbhd of I by (left) multiplication by A.