HW4 #4

V = real vector space w. basis $e_1 \dots e_n$

$V \otimes \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$ = complex vector space w. basis $e_1 \dots e_n$

$\dim_{\mathbb{C}}(V \otimes \mathbb{C}) = n$

$\dim_{\mathbb{R}}(V \otimes \mathbb{C}) = 2n$.

$\mathfrak{g}$ = real Lie algebra with basis $e_1 \dots e_n$

$[e_i, e_j] = \sum c_{ij}^k e_k \quad c_{ij}^k \in \mathbb{R}$

$\mathfrak{g} \otimes \mathbb{C}$ = complex Lie algebra with same basis and same commutation rels.

Note: $[\,,\,]$ on $\mathfrak{g}$ is $\mathbb{R}$-bilinear
$[\,,\,]$ on $\mathfrak{g} \otimes \mathbb{C}$ is $\mathbb{C}$-bilinear.

Ex $sl_n(\mathbb{R}) = \{X \in Mat(n \times n, \mathbb{R}) : Tr(X) = 0\}$

$sl_n(\mathbb{R}) \otimes \mathbb{C} = \{X \in Mat(n \times n, \mathbb{C}) : Tr(X) = 0\} = sl_n(\mathbb{C})$.

Ex $so_n(\mathbb{R}) \otimes \mathbb{C} = so_n(\mathbb{C})$

HW: $so_3(\mathbb{R}) \otimes \mathbb{C} \simeq su_2 \otimes \mathbb{C} \simeq sl_2(\mathbb{C})$.
($\Rightarrow so_3(\mathbb{C}) \simeq sl_2(\mathbb{C})$ as Lie complex algebras).

Fact · $su_n \otimes \mathbb{C} \simeq sl_n(\mathbb{C})$
· $u_n \otimes \mathbb{C} \simeq gl_n(\mathbb{C})$ Note: $\dim u_n = n^2$ $\dim_{\mathbb{C}} gl_n(\mathbb{C}) = n^2$

Proof $u_n = \{X : X^* + X = 0\}$.

Suppose $X^* + X = 0$

Then $(iX)^* = -iX^*$, if $X^* = -X$ then $(iX)^* = iX$

$L = \{Y : Y^* = Y\}$

Claim: $\cdot\ X \in \mathfrak{u}_n \iff iX \in L$ ✓

- $\mathfrak{u}_n \cap L = \{0\}$ $\qquad X^* = -X$ and $X^* = X \Rightarrow X = 0$. ✓
- $\mathfrak{u}_n \oplus L = \mathfrak{gl}_n(\mathbb{C})$

$M \in \mathfrak{gl}_n(\mathbb{C})$, define $X = \dfrac{M - M^*}{2}$ $\quad Y = \dfrac{M + M^*}{2}$

Then $\underset{X \in \mathfrak{u}_n}{X^* = -X}$, $\underset{Y \in L}{Y^* = Y}$, $X + Y = M$.

$\Rightarrow \mathfrak{u}_n \otimes \mathbb{C} = \mathfrak{u}_n \oplus i\mathfrak{u}_n = \mathfrak{u}_n \oplus L = \mathfrak{gl}_n(\mathbb{C})$ ∎

Another way: $\mathfrak{su}_2 \subset \mathfrak{sl}_2(\mathbb{C})$ $\qquad$ ($\mathfrak{so}_3(\mathbb{R}) \simeq \mathfrak{su}_2$)

Need to prove that the $\mathbb{R}$-basis of $\mathfrak{su}(2)$ can serve as a $\mathbb{C}$-basis for $\mathfrak{sl}_2(\mathbb{C})$.

Thm (3.42) G = matrix Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$

There exists an $0 < \varepsilon < \log 2$ such that:

$V_\varepsilon = \{X : \|X\| < \varepsilon\}$, $U_\varepsilon = \exp(V_\varepsilon)$

$A \in U_\varepsilon \cap G \iff \log A \in V_\varepsilon \cap \mathfrak{g}$.

Checked earlier: $\|X\| < \log 2 \Downarrow \|e^X - I\| < 1 \Rightarrow \log(e^X)$ is defined

Proof: Skip, see the book.

Thm If G = matrix Lie group then G is a smooth manifold and $\mathfrak{g} = T_I G$ ← tangent space at I.

Proof: We need to present a coordinate chart near every point $A \in G$.

① Near I, we can take the chart $U_\varepsilon \cap G$ from Thm 3.42.

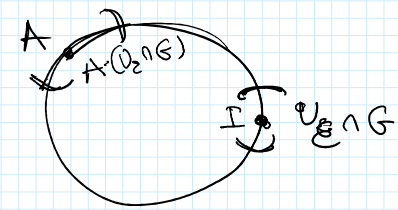
$$U_\varepsilon \cap G \underset{\exp}{\overset{\log}{\rightleftarrows}} V_\varepsilon \cap \mathfrak{g}$$

By our choice of ε + Thm 3.42, Log, Exp are well defined.

By our choice of $\varepsilon^{\exp}$ + Thm 3.42 Log, Exp are well defined, continuous & inverse to each other $\Rightarrow$ homeomorphisms.
We identified $U_\varepsilon \cap G$ with an open subset of the vector space $\mathfrak{g}$.

(2) Pick $A \in G$, need to construct coordinate chart near A

$$A \cdot (U_\varepsilon \cap G) \subset G$$



left multiplication $A\cdot(-) : G \longrightarrow G$ continuous map.

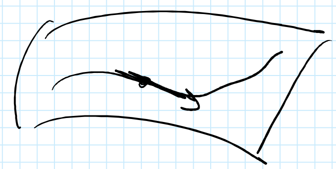
<u>Note</u>: every element of that neighborhood can be written as $A \cdot e^X$, $X \in V_\varepsilon \cap \mathfrak{g}$. (*)

(3) Conclusion: G is a manifold with such charts & local coordinates

(4) M = manifold

$\gamma(t) : \mathbb{R} \longrightarrow M$ smooth path

$\gamma'(t)$ is a tangent vector to M at point $\gamma(t)$!



(*) Transition functions

$A \cdot e^X = B \cdot e^Y$

$B^{-1}A \cdot e^X = e^Y$

$\mathrm{Log}(B^{-1}A \cdot e^X) = Y$.

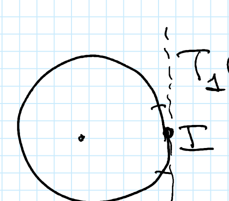
Exp, Log smooth $\Rightarrow$ Y is a smooth fn. of X

$A(t) = e^{tX}$ = path in G

$$\Rightarrow A'(0) = \frac{d}{dt} e^{tX}\Big|_{t=0} = X \in T_I G$$

$\mathfrak{g} \subset T_I G$ because $\underset{\text{as v. space}}{\dim \mathfrak{g}} = \underset{\text{as mfd}}{\dim G} = \dim T_I G$ we get $\mathfrak{g} = T_I G$.

Therefore $\boxed{\mathfrak{g} = T_I G}$



$T_1 U(1) = \{ia : a \in \mathbb{R}\} = \mathrm{Lie}\, U(1)$

(5) How to describe the tangent space at other point A?

$$\frac{d}{dt} A \cdot e^{tX}\Big|_{t=0} = A \cdot X \Rightarrow \boxed{T_A G = A \cdot \mathfrak{g}}$$

Rmk G is parallelizable, that is, the tangent bundle to G is trivial.

Notes: ① $G_0 \subset G$ connected component of I

then $\mathrm{Lie}\, G_0 = \mathrm{Lie}\, G$.

Idea: $e^{tX} \in G_0$ since it is a continuous path to $e^{0X} = I$.

② If G is connected then it is generated by a neighborhood of I, in particular, any element of G can be written as $e^{X_1} \cdot e^{X_2} \cdot \ldots \cdot e^{X_k}$ for some k.