Recap $G =$ matrix Lie group

Last time: • there's a neighborhood U_ε of I such that every element of U_ε has the form e^X, $X \in Lie(G)$

• $G =$ smooth manifold
$Lie(G) = T_I G$.

Thm $G =$ connected matrix Lie group
Any open neighborhood of I generates G

Corollary Any element of G can be written as
$$e^{X_1} \cdots e^{X_k} \text{ for some } X_1 \ldots X_k \in Lie(G)$$

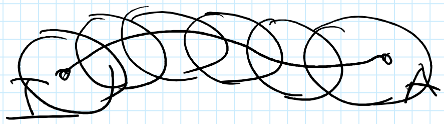
Proof: $U \ni I$ open subset $S = \{u_1 \ldots u_k : u_i \in U\}$
Without loss of generality, assume $U = U^{-1}$
(if not, consider $U' = U \cap U^{-1}$)

• S open in G
(given $u_1 \ldots u_k$, take open nbhd of $u_i \subset U$
this gives nbhd of the product)

• $G \setminus S$ open in G: suppose $A \in G \setminus S$
$A \cdot U =$ open subset containing A
let's prove that it does not intersect S. Indeed, if
$B \in S \cap (A \cdot U)$, we can write $B = u_1 \ldots u_k$
also $B = Au$, then $A = Bu^{-1} = u_1 \ldots u_k \cdot u^{-1}$
therefore $A \in S$. Contradiction.

Therefore $G \setminus S$ is open $\Rightarrow$ S closed & open & non-empty
Since G is connected, we get $G = S$. ∎

<u>Alternative proof</u>



G connected manifold
$\Downarrow$
path connected.

Choose a path connecting I and A
For any point B on the path, choose a nbhd $B \cdot U$
Since the path is compact, choose finitely many of these. ...

<u>Homomorphisms</u> <u>Def:</u> A Lie group homomorphism $\Phi : G \to H$ is a <u>continuous</u> <u>homomorphism</u>.

<u>Def</u> $\mathfrak{g}, \mathfrak{h}$ = Lie algebras. A Lie algebra homomorphism is a <u>linear map</u> $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$[\varphi(X), \varphi(Y)] = \varphi([X, Y]) \text{ for all } X, Y.$$

<u>Thm (3.28)</u> $\Phi : G \to H$ Lie group homomorphism
Then there exists a unique Lie algebra homomorphism
$\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$
such that

① $\Phi(e^X) = e^{\varphi(X)}$ for all $X \in \mathfrak{g}$

② $\varphi(X) = \frac{d}{dt} \Phi(e^{tX}) \Big|_{t=0}$

<u>Thm</u> (3.50) If Φ is a continuous Lie group homomorphism then Φ is smooth!

Proof Suppose $A \in G$, then every element in some nbhd of A can be written as Ae^X, $X \in \mathfrak{g}$.

$$\Phi(Ae^X) = \Phi(A)\cdot\Phi(e^X) = \Phi(A)\cdot e^{\varphi(X)}$$

X gives local coords near A.

$X \to \varphi(X)$ linear map, so smooth.

$X \to e^{\varphi(X)}$ is smooth since exp is smooth.

Rmk $\Phi : G \to H$ Lie group homomorphism

smooth, $\Phi(I) = I \Rightarrow$ linear map of tangent spaces

$$d\Phi : \underset{\underset{\text{Lie } G}{\parallel}}{T_I G} \longrightarrow T_{\Phi(I)} H = \underset{\underset{\text{Lie } H}{\parallel}}{T_I H}$$

In fact, $d\Phi = \varphi$ is a Lie algebra homomorphism.

Fact $\Phi : G \to H$ $\quad \underset{\underset{d\Phi}{\parallel}}{\varphi} : \text{Lie}(G) \to \text{Lie}(H)$

Suppose H is connected and φ is surjective. Then Φ is surjective.

Proof Pick $B \in H$, by Thm above we can write

$$B = e^{Y_1} \cdot \ldots \cdot e^{Y_k} \text{ for some } Y_1 \ldots Y_k \in \text{Lie}(H).$$

Since φ is surjective, we can find $X_1 \ldots X_k \in \text{Lie}(G)$ such that $Y_i = \varphi(X_i)$, then

$$e^{Y_1} \cdots e^{Y_k} = e^{\varphi(X_1)} e^{\varphi(X_2)} \cdots e^{\varphi(X_k)} =$$

$$= \Phi(e^{X_1})\Phi(e^{X_2}) \cdot \ldots \cdot \Phi(e^{X_k}) =$$

$$= \Phi(e^{X_1} \cdots e^{X_k}).$$

<u>Application</u> $\Phi : SU(2) \longrightarrow SO(3)$)

Recall that there's a 3d real vector space

$\mathcal{U} = \{X : X = X^*, Tr(X) = 0\} \leftarrow$ 2x2 matrices.

$Lie(SU(2)) = \{X : X = -X^*, Tr(X) = 0\}$

So $\mathcal{U} = i\, Lie(SU(2)) \simeq Lie(SU(2))$

1) Since $SU(2)$ acts on $Lie(SU(2))$ by conjugation

this is the same representation! $\boxed{A \in G,\ X \in Lie\, G \Rightarrow AXA^{-1} \in Lie\, G}$

2) There's a symmetric positive definite bilinear form

$Tr(XY)$ on $\mathcal{U}$ / $Lie(SU(2))$ } Killing form.

which is preserved by this action

3) This gives a homomorphism $\Phi : SU(2) \longrightarrow O(3)$

Since $SU(2)$ is connected, $\Phi : SU(2) \longrightarrow SO(3)$.
image is connected.

4) $Ker = \{\pm I\}$. To check Φ is surjective onto $SO(3)$,

it is sufficient to prove $d\Phi : \underset{\underset{Lie(SU(2))}{\shortparallel}}{su(2)} \xrightarrow{\sim} so(3)$

is surjective.

this is the iso from HW4.

<u>Claim</u> $d\Phi(X) = [X, -]$

understood as an operator from

$Lie(SU(2)) \longrightarrow Lie(SU(2))$.

$\dim su(2) = 3 \Longrightarrow$ this gives a 3×3 matrix.