Lecture 13 (4/27)

Wednesday, April 27, 2022 1:07 PM

Def A Lie group representation is a vector space V and a Lie group homomorphism $\Phi : G \longrightarrow GL(V)$

(recall that Φ is continuous and, in fact, smooth)

Def A Lie algebra representation is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = n\times n \text{ matrices } (n = \dim V)$$

Fact If $\Phi : G \longrightarrow GL(V)$ Lie group representation

$\Rightarrow d\Phi : \mathrm{Lie}(G) \longrightarrow \mathfrak{gl}(V)$ Lie algebra representation

Ex G = matrix Lie group, always has <u>defining</u> representation.

$$G \subset GL(n) \Longrightarrow G \xrightarrow{id} GL(n) \text{ repres.}$$

Ex $G = GL(n)$ or $SL(n)$ has n-dim. defining repres.

Def The <u>adjoint</u> representation of G:

$$V = \mathrm{Lie}(G), \quad \Phi : G \to GL(V)$$

$\Phi = \mathrm{Ad}$, $\Phi(A)\cdot X = \mathrm{Ad}_A(X)$

$A \in G$, $X \in \mathrm{Lie}\,G$, define $\Phi(A)\cdot X = \underline{AXA^{-1}} \in \mathrm{Lie}(G)$ linear in X

$$\Phi(AB)\cdot X = (AB)X\cdot(AB)^{-1} = ABXB^{-1}A^{-1} = \Phi(A)\left(\Phi(B)(X)\right)$$

Def The adjoint representation of $\mathfrak{g}$:

ad

<u>**Def**</u> The adjoint representation of $\mathfrak{g}$:

$$V = \mathfrak{g}, \quad \varphi \overset{=\mathrm{ad}}{} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$X \in \mathfrak{g},\ Y \in \mathfrak{g} \qquad \underbrace{\varphi(X) \cdot Y}_{\mathrm{ad}_X(Y)} = [X, Y]$$
$$= XY - YX$$

This is a Lie algebra representation!

$$[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]} \longleftarrow \text{need to check}$$

Apply this to $Z \in \mathfrak{g}$:

$$\left.\begin{aligned} \mathrm{ad}_{[X,Y]}(Z) &= [[X,Y], Z] \\ \mathrm{ad}_X \mathrm{ad}_Y (Z) &= [X, [Y, Z]] \\ \mathrm{ad}_Y \mathrm{ad}_X (Z) &= [Y, [X, Z]] \end{aligned}\right\} \begin{aligned} &\text{by Jacobi identity} \\ &\mathrm{ad}_X \mathrm{ad}_Y(Z) - \mathrm{ad}_Y \mathrm{ad}_X(Z) \\ &= \mathrm{ad}_{[X,Y]}(Z). \end{aligned}$$

<u>**Fact**</u> G = matrix Lie group, $d(\underbrace{\mathrm{Ad}}_{\text{adjoint for } G}) = \underbrace{\mathrm{ad}}_{\text{adjoint for } \mathrm{Lie}(G)}$.

$su(2)$ $\quad X^* = -X$, $\quad \mathrm{Tr}(X) = 0$

$$\begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}$$

basis: $e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$[e_1, e_2] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2e_3$$

$[e_1, e_3] = -2e_2$

$[e_2, e_3] = 2e_1$

How to compute the adjoint representation for $su(2)$?

ad_{e_1}: $e_1 \longrightarrow [e_1, e_1] = 0$

$e_2 \longrightarrow [e_1, e_2] = 2e_3$

$e_3 \longrightarrow [e_1, e_3] = -2e_2$

$$ad_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

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ad_{e_2}: $e_1 \longrightarrow [e_2, e_1] = -2e_3$

$e_2 \longrightarrow [e_2, e_2] = 0$

$e_3 \longrightarrow [e_2, e_3] = 2e_1$

$$ad_{e_2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

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ad_{e_3}: $e_1 \longrightarrow [e_3, e_1] = 2e_2$

$e_2 \longrightarrow [e_3, e_2] = -2e_1$

$e_3 \longrightarrow [e_3, e_3] = 0$

$$ad_{e_3} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Note</u>: this is the isomorphism $su(2) \longrightarrow so(3)$

Lie algebra homomorphism $su(2) \longrightarrow gl(3)$

image $= so(3)$.

<u>Fact</u> Suppose $\varphi : \mathfrak{g} \longrightarrow gl(n, \mathbb{C})$

is a <u>complex</u> Lie algebra representation.

Then we have a representation

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow gl(n, \mathbb{C})$$

[$\mathfrak{g}$ = real Lie alg]

by $\mathbb{C}$-linear extension of φ.

Cor $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{su}(2)$, $\mathfrak{so}(3,\mathbb{R})$, $\mathfrak{so}(3,\mathbb{C})$ have the <u>same</u> complex Lie algebra representations!

$$\mathfrak{sl}_2(\mathbb{R})\otimes\mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{su}(2)\otimes\mathbb{C} \simeq \mathfrak{so}(3,\mathbb{R})\otimes\mathbb{C} \simeq \mathfrak{so}(3,\mathbb{C})$$

Warning Although $\mathfrak{sl}_2(\mathbb{R})\otimes\mathbb{C} \simeq \mathfrak{su}(2)\otimes\mathbb{C}$

$\mathfrak{sl}_2(\mathbb{R}) \not\simeq \mathfrak{su}(2)$ as Lie algebras.

Thm (Sec. 5) Suppose G, H are matrix Lie groups

G is connected & <u>simply connected</u> $(\pi_1(G) = \{e\})$

Suppose we have a Lie algebra homomorphism

$$\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$$

Then there exists a unique homomorphism

$$\Phi : G \to H \quad \text{such that} \quad d\Phi = \varphi.$$

Cor Suppose $\varphi : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation.

$$\varphi' : \mathfrak{su}(n) \to \mathfrak{gl}(V)$$

Then there exist Lie group representations

$$\Phi : SL_n(\mathbb{C}) \to GL(V) \qquad \Phi' : SU(n) \to GL(V)$$

such that $d\Phi = \varphi$, $d\Phi' = \varphi'$.

Proof Cor: $\pi_1(SU(n)) = \pi_1(SL_n(\mathbb{C})) = \{e\}$.

Idea of proof of Thm: We are given $\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$

Define $\Phi : U_\varepsilon(I) \to U_{\varepsilon'}(I)$

Define $\Phi : U_\varepsilon(I) \longrightarrow U_{\varepsilon'}(I)$

neighborhood of I

by $\Phi(e^X) = e^{\varphi(X)}$

Hard: this is a "local homomorphism"

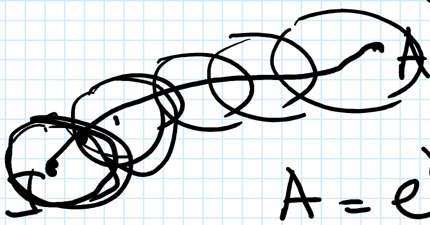
$$\Phi(e^{X_1}e^{X_2}) = \Phi(e^{X_1})\,\Phi(e^{X_2}).$$

Campbell-Baker-Hausdorff.

$$e^{X_1}e^{X_2} = e^{Y}, \quad Y = X_1 + X_2 + \tfrac{1}{2}[X_1, X_2] + \dots$$

all terms involve commutators.

Continue Φ along paths to any point of G:



$A(t)$: path from I to A

cover it by translates of U_ε

$A = e^{X_1} \dots e^{X_k}$ ← depends on a path!

$\Phi(A) = e^{\varphi(X_1)} \dots e^{\varphi(X_k)}$.

Since $\pi_1(G) = \{e\}$, any two paths are homotopic and one can prove that this def. of Φ does not depend on the path.