Lecture 14 (4/29)
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Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

Lie algebra

$\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H,E] = 2E \quad [H,F] = -2F \quad [E,F] = H. \quad (*)$$

A rep. of $\mathfrak{sl}_2(\mathbb{C})$ is a vector space V over $\mathbb{C}$ with operators E, F, H satisfying $(*)$.

Key example: Verma module

Start from a vector $v = v_0$, assume $\left.\begin{array}{l} Ev = 0 \\ Hv = \lambda v \end{array}\right]$

$$\begin{tikzcd} 0 & v_0 \arrow[l, bend right, "E"'] \arrow[r, bend left, "F"] & v_1 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & v_2 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & v_3 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & \cdots \cdots \arrow[l, bend left, "E"] \end{tikzcd}$$

$$v_0 = v, \quad v_1 = Fv, \quad v_2 = F^2 v, \quad v_3 = F^3 v$$

$$\xleftarrow{E} \qquad \xrightarrow{F}$$

Define an infinite sequence of vectors $v_0, v_1, v_2, \dots$
$v_k = F^k v_0$

$$\Delta(\lambda) = \underbrace{\mathrm{Span}}_{\text{finite linear combinations}}(v_0, v_1, v_2, \dots)$$

Claim We can define a representation of $\mathfrak{sl}_2$ on $\Delta(\lambda)$.

How to compute an action of H, E on $\Delta(\lambda)$?

Know $Ev_0 = 0$, $Hv_0 = \lambda v_0$

$$Ev_1 = E(Fv_0) = EF(v_0) = FE(v_0) + [E,F](v_0) = FE(v_0) + Hv_0 = 0 + \lambda v_0 = \lambda v_0.$$

$= FE(v_0) + [E,F](v_0)$

$= FE(v_0) + Hv_0 = 0 + \lambda v_0 = \underline{\lambda v_0}$.

$Hv_1 = H(Fv_0) = HFv_0 = FHv_0 + [H,F]v_0$

$= F(\lambda v_0) - 2Fv_0 = (\lambda - 2)Fv_0 = (\lambda-2)v_1$

So v_1 is also an eigenvector for H with eigenvalue $\lambda - 2$.

$Ev_2 = E(Fv_1) = FEv_1 + [E,F]v_1 = F\underbrace{Ev_1} + \underbrace{Hv_1}$

$= F(\lambda v_0) + (\lambda-2)v_1$ just computed these!

$= \lambda F v_0 + (\lambda-2)v_1 = (\lambda + \lambda - 2)v_1$

$= (2\lambda - 2)v_1$

$Hv_2 = HF(v_1) = FH(v_1) + [H,F]v_1 =$

$= F(\lambda-2)v_1 - 2Fv_1 = (\lambda-4)Fv_1 = (\lambda-4)v_2$.

So v_2 is also an eigenvector for H with eigenvalue $\lambda - 4$.

__Fact:__ $Hv_k = (\lambda - 2k)v_k$, so v_k are eigenbasis of $\Delta(\lambda)$

$Ev_k = \underline{k(\lambda - k + 1)}v_{k-1}$

$Fv_k = v_{k+1}$

__Proof:__ __HW 5__ to do. (induction in k)

__$\lambda = 3$__

$$\begin{tikzcd}
v_0 \arrow[r, bend left, "F"] & v_1 \arrow[l, bend left, "3"] \arrow[r, bend left, "F"] & v_2 \arrow[l, bend left, "4"] \arrow[r, bend left, "F"] & v_3 \arrow[l, bend left, "3"] \arrow[r, bend left, "F"] & v_4 \arrow[l, bend left, "0"] \arrow[r, bend left] & v_5 \arrow[l, bend left, "-5"] \cdots
\end{tikzcd}$$

$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$\lambda=3$	$\lambda=3$	$\lambda=3$	$\lambda=3$	$\lambda=3$

__Note__ that in this case $\mathrm{Span}(v_4, v_5, v_6, \ldots) = R$ is a <u>subrepresentation/invariant subspace</u>!

Clearly preserved by H, F

Can consider the quotient $L(3) = \Delta(3)/R$

$$L(3):\quad \begin{tikzcd} 0 & v_0 \arrow[l, "E=0"'] \arrow[r, bend left, "F"] & v_1 \arrow[l, bend left, "3"] \arrow[r, bend left, "F"] & v_2 \arrow[l, bend left, "4"] \arrow[r, bend left, "F"] & v_3 \arrow[l, bend left, "3"] \arrow[r, bend left, "F"] & 0 \end{tikzcd}$$

$Hv_0 = 3v_0 \quad Hv_2 = -v_2$

$Hv_1 = v_1 \quad Hv_3 = -3v_3$

We have constructed a 4-dimensional representation $L(3)$ of the Lie algebra sl_2.

<u>Thm</u> (1) If $\lambda = n$ is a nonnegative integer then $\Delta(n)$ has an invariant subspace R_n spanned by $v_{n+1}, v_{n+2}, \ldots$

The quotient $\underbrace{\Delta(n)/R_n}_{L(n)=}$ is a finite dimensional representation of sl_2 of dimension $n+1$.

(2) If $\lambda = n$ is a nonnegative integer then $L(n)$ is an <u>irreducible</u> representation of sl_2

(3) If λ is not a nonnegative integer then $\Delta(\lambda)$ is irreducible

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<u>Proof</u> (3) Suppose $V \subset \Delta(\lambda)$ is an invariant subspace

$0 \neq u = \sum \alpha_i v_i \qquad$ finitely many α_i

Suppose that $m = \max\{i : \alpha_i \neq 0\}$

We claim that $E^m u \neq 0$ and $E^m u$ is a nonzero multiple of v_0.

For $i < m$ we have $E^m v_i = 0$

Also $Ev_k = k(\lambda - k + 1) v_{k-1}$ for all k

Since λ is <u>not</u> a nonnegative integer, $\lambda - k + 1 \neq 0$ for all k.

$Ev_k = (\text{nonzero coef}) v_{k-1}$

$E^m v_m = (\text{nonzero coef}) v_0$

$u = \underset{\text{span}(v_i,\, i<m)}{\underbrace{\dots}} + \alpha_m v_m \Rightarrow E^m u = \alpha_m \cdot (\text{nonzero cf}) v_0$

Therefore V contains $u, E^m u; v_0 \Rightarrow$ contains all $F^k v_0 = v_k$

$\Rightarrow V = \Delta(\lambda)$.

① Clear, same as $\lambda = 3$

② Use same idea as ③ to prove that R_n is the only proper invariant subspace of $\Delta(n)$ — finish in HW.