

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

\mathfrak{sl}_2 algebra

$\mathfrak{sl}_2(\mathbb{C})$ has a basis

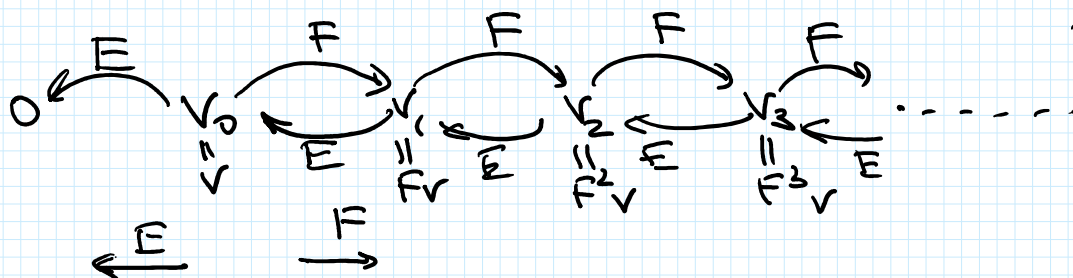
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H, E] = 2E \quad [H, F] = -2F \quad [E, F] = H. \quad (*)$$

A rep. of $\mathfrak{sl}_2(\mathbb{C})$ is a vector space V with operators over \mathbb{C} E, F, H satisfying $(*)$.

Key example: Verma module

Start from a vector $v = v_0$, assume $E v = 0$
 $H v = \lambda v$



Define an infinite sequence of vectors v_0, v_1, v_2, \dots
 $v_k = F^k v_0$

$$\Delta(\lambda) = \text{Span}(v_0, v_1, v_2, \dots)$$

finite linear combinations.

Claim We can define a representation of \mathfrak{sl}_2 on $\Delta(\lambda)$.

How to compute an action of H, E on $\Delta(\lambda)$?

$$\text{Know } \begin{array}{l} E v_0 = 0 \\ H v_0 = \lambda v_0 \end{array}$$

$$\begin{aligned} E v_1 &= E(F v_0) = EF(v_0) = \\ &= FE(v_0) + [E, F](v_0) \\ &= FE(v_0) + H v_0 = 0 + \lambda v_0 = \lambda v_0. \end{aligned}$$

$$= FE(v_0) + Hv_0 = 0 + \lambda v_0 = \underline{\lambda v_0}.$$

$$\begin{aligned} H v_1 &= H(F v_0) = HF v_0 = FH v_0 + [H, F] v_0 \\ &= F(\lambda v_0) - 2F v_0 = (\lambda - 2) F v_0 = (\lambda - 2) v_1 \end{aligned}$$

So v_1 is also an eigenvector for H with eigenvalue $\lambda - 2$.

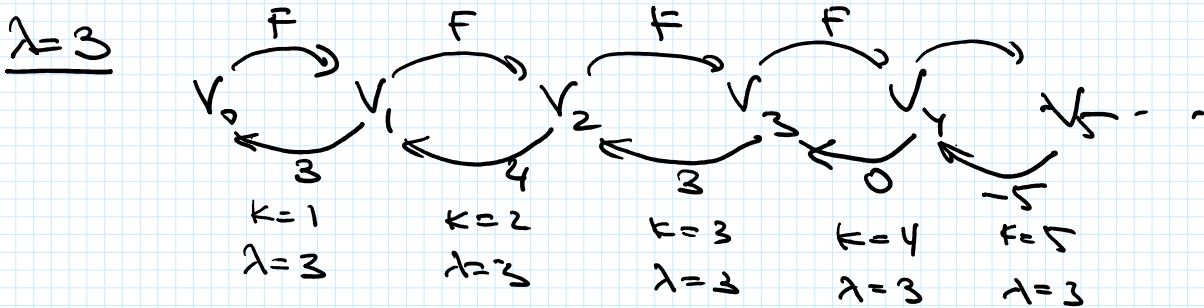
$$\begin{aligned} E v_2 &= E(F v_1) = FE v_1 + [E, F] v_1 = \underbrace{F E v_1}_{\text{just computed these!}} + \underbrace{H v_1}_{\text{just computed these!}} \\ &= F(\lambda v_0) + (\lambda - 2) v_1 \\ &= \lambda F v_0 + (\lambda - 2) v_1 = (\lambda + \lambda - 2) v_1 \\ &= (2\lambda - 2) v_1 \end{aligned}$$

$$\begin{aligned} H v_2 &= HF(v_1) = FH(v_1) + [H, F] v_1 = \\ &= F(\lambda - 2) v_1 - 2F v_1 = (\lambda - 4) F v_1 = (\lambda - 4) v_2. \end{aligned}$$

So v_2 is also an eigenvector for H with eigenvalue $\lambda - 4$.

Fact: $H v_k = (\lambda - 2k) v_k$, so v_k are eigenbasis of $\Delta(\lambda)$

Proof:
HW5
 to do.
 (induction in k)

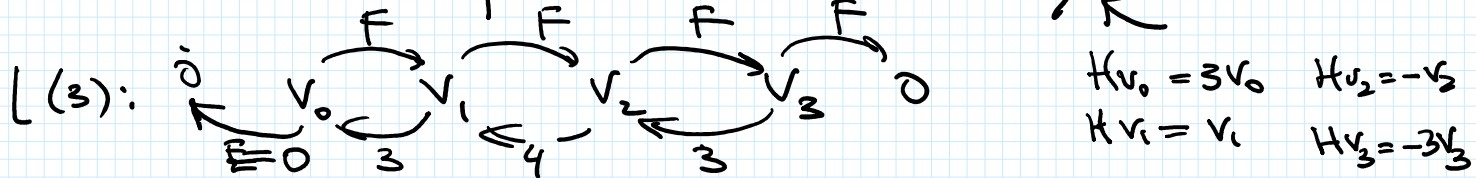
$$\begin{aligned} E v_k &= k(\lambda - k + 1) v_{k-1} \\ F v_k &= v_{k+1} \end{aligned}$$


Note that in this case $\text{Span}(v_4, v_5, v_0, \dots) = R$ is a subrepresentation invariant subspace!

Π ...

Clearly preserved by H, F

Can consider the quotient $L(3) = \Delta(3) / R$



We have constructed a 4-dimensional representation $L(3)$ of the Lie algebra \mathfrak{sl}_2 .

Thm ① If λ is a nonnegative integer then $\Delta(\lambda)$ has an invariant subspace R_λ spanned by $v_{\lambda+1}, v_{\lambda+2}, \dots$

The quotient $\Delta(\lambda) / R_\lambda$ is a finite dimensional representation of \mathfrak{sl}_2 of dimension $\lambda+1$.

② If $\lambda = n$ is a nonnegative integer then $L(n)$ is an irreducible representation of \mathfrak{sl}_2

③ If λ is not a nonnegative integer then $\Delta(\lambda)$ is irreducible

Proof ③ Suppose $V \subset \Delta(\lambda)$ is an invariant subspace $0 \neq v = \sum \alpha_i v_i$ finitely many α_i

Suppose that $m = \max\{i : \alpha_i \neq 0\}$

We claim that $E^m v \neq 0$ and $E^m v$ is a nonzero multiple of v_0 .

For $i < m$ we have $E^m v_i = 0$

Also $EV_k = k(\lambda - k + 1)v_{k-1}$ for all k

Since λ is not a nonnegative integer, $\lambda - k + 1 \neq 0$
for all k .

$$EV_k = (\text{nonzero coef}) v_{k-1}$$

$$E^m v_m = (\text{nonzero coef}) v_0$$

$$u = \underbrace{\dots + d_m v_m}_{\text{Span}(v_i, i < m)} \Rightarrow E^m u = d_m \cdot (\text{nonzero coef}) v_0$$

Therefore \mathcal{V} contains $u, E^m u, v_0 \Rightarrow$ contains all $F_{v_0}^k = v_k$
 $\Rightarrow \mathcal{V} = \Delta(\lambda)$.

① Clear, same as $\lambda = 3$

② Use same idea as ③ to prove that

R_n is the only proper invariant subspace of $\Delta(n)$

} finish in HW.