Recap: • Verma module $\Delta(\lambda)$

$$v_0,\ v_1 = Fv_0,\ v_2 = F^2 v_0,\ \ldots$$

$$Ev_0 = 0 \quad Hv_0 = \lambda v_0$$

• If λ not a nonnegative integer, $\Delta(\lambda)$ irreducible

• If $\lambda = n$ is a nonnegative integer,

$$L(n) = \Delta(n) / \operatorname{Span}(v_{n+1}, v_{n+1}, \ldots)$$

is a finite dim irreducible representation.

Thm 1 (today) Any finite-dimensional irreducible (complex) representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $L(n)$ for some n.

Thm 2 (proof Wed) Any finite-dim representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to a direct sum of irreducibles

$$V = \bigoplus L(n_i) \quad \text{for some } n_i \geq 0.$$

Proof of Thm 1: Suppose V is an irreducible finite-dim complex representation of $\mathfrak{sl}_2(\mathbb{C})$.

Step 1 Let $v \in V$ be an eigenvector for H with some eigenvalue μ, so $Hv = \mu v$

If $Ev = 0$, stop for now.

If $Ev \neq 0$, observe that $H(Ev) = HEv = EHv + [H,E]v$

$$= E(\mu v) + 2Ev = (\mu + 2)Ev$$

So Ev is an eigenvector for H with eigenvalue $\mu+2$.

If $E^2v=0$, stop for now, otherwise consider the sequence of vectors

$$\begin{array}{lccccc} & v, & Ev, & E^2v, & \dots & E^kv, \dots \\ \text{eigenvalue for } H \longrightarrow & \mu & \mu+2 & \mu+4 & & \mu+2k \end{array}$$

Since V is finite dimensional, eventually $E^kv=0$ for some k

Therefore we found a vector $u=E^{k-1}v$ such that $\boxed{Eu=0,\ Hu=\lambda u \text{ for some } \lambda}$ ← highest weight vector.

Step 2 Given such u, we can construct a morphism of representations $\Delta(\lambda) \xrightarrow{\varphi} V$

$$\begin{array}{ccc} v_0 & \longrightarrow & u \\ v_1 & \longrightarrow & Fu \\ \vdots & & \vdots \\ v_k & \longrightarrow & F^ku \\ \vdots & & \vdots \end{array}$$

Similarly to last lecture the map $\varphi: \Delta(\lambda) \to V$ agrees with the action of sl_2.

Step 3 Let's study $\operatorname{Ker}\varphi \subset \Delta(\lambda)$
$\operatorname{Im}\varphi \subset V$.

$\operatorname{Ker}(\varphi) = sl_2$ invt subspace of $\Delta(\lambda)$

If λ is not a nonnegative integer, $\Delta(\lambda)$ is irreducible

$\Rightarrow \operatorname{Ker}(\varphi) = \{0\}$.

[note $\operatorname{Ker}\varphi \neq \Delta(\lambda)$ since $\varphi(v_0)=u$ and $v_0 \notin \operatorname{Ker}$]

$\Rightarrow \varphi$ injective, $\dim \operatorname{Im}\varphi = \infty$

Contradiction!

Therefore $\boxed{\lambda \text{ is a nonnegative integer}}$

$\Rightarrow \operatorname{Ker}(\varphi) = \{0\}$ or $\operatorname{Ker}\varphi = \operatorname{Span}(v_{n+1}, v_{n+2}, \dots)$

only two sl_2-invt. subspaces!

only two $\mathfrak{sl}_2$-invt. subspaces!

If $\mathrm{Ker}(\varphi) = \{0\}$, again
$\mathrm{Im}(\varphi)$ infinite-dim, contradiction

If $\mathrm{Ker}(\varphi) = \mathrm{Span}(v_{n+1}, v_{n+2}, \dots)$ then

$$\mathrm{Im}(\varphi) \cong \frac{\Delta(\lambda)}{\mathrm{Span}(v_{n+1}, \dots)} = \boxed{L(\lambda)}$$

Step 4 Since V is irreducible and $\mathrm{Im}\,\varphi$ is an $\mathfrak{sl}_2$-invt subspace, we get $V = \mathrm{Im}\,\varphi \cong L(\lambda)$.

Cor (Thm 1 + Thm 2) V = any f.d. representation of $\mathfrak{sl}_2$
$\Rightarrow$ H diagonalizable, all eigenvalues of H are integers!

Proof: This is true for each $L(n)$, hence true for $\bigoplus L(n_i)$
$\Rightarrow$ E, F nilpotent on V.

Define $a_i = \dim \{v \in V : Hv = iv\}$ $\quad i \in \mathbb{Z}$
eigenspace for H w. eigenvalue i

Cor $a_i = a_{-i}$, so the sequence of a_i is symmetric!

Proof: Sufficient to prove this for $L(n)$, then it holds for $\bigoplus L(n_i)$.

$L(n)$	v_0	v_1	v_2	$\dots$	v_n
H :	n	$n-2$	$n-4$	$\dots$	$-n = n-2n$

This is symmetric around 0!

$$a_i \text{ for } L(n) = \begin{cases} 1, & |i| \le n, \quad i \equiv n \bmod 2 \\ 0, & \text{otherwise.} \end{cases}$$

Ex V = representation of sl_2,

i	3	2	1	0	-1	-2	-3
a_i	1	2	2	3	2	2	1

multiplicities of eigenvalues for H

Claim: We can look at these numbers and decompose $V = \bigoplus L(n_i)$

i	3	2	1	0	-1	-2	-3
a_i	1	2	2	3	2	2	1
$L(3)$	1		1		1		1
$L(2)$		1		1		1	
$L(1)$			1		1		
$L(0)$				1			

eigenvalues ≤ 3
$\Rightarrow$ all $n_i \leq 3$

$$V = L(3) + 2L(2) + L(1) + L(0)$$

The multiplicities of $L(n_i)$ are uniquely determined by a_i = multiplicities of eigenvalues for H.

Character: V = representation of sl_2

$$\operatorname{ch} V = \sum x^i a_i = \text{generating function for } a_i\text{'s}$$

Laurent polynomial in x

Clearly $\operatorname{ch}(V_1 \oplus V_2) = \operatorname{ch} V_1 + \operatorname{ch} V_2$

Ex $\operatorname{ch} L(n) = x^n + x^{n-2} + x^{n-4} + \dots + x^{-n}$

$$= \boxed{\frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}}$$

geometric sequence.

$\operatorname{ch} L(0) = 1$ $\qquad$ $\operatorname{ch} L(2) = x^2 + 1 + x^{-2}$

$\operatorname{ch} L(1) = x + x^{-1}$ $\qquad$ $\operatorname{ch} L(3) = x^3 + x + x^{-1} + x^{-3}$ $\Leftarrow$ ....

Above, we can just decompose

$$\operatorname{ch} V = x^3 + 2x^2 + 2x + 3 + 2x^{-1} + 2x^{-2} + x^{-3}$$

as a linear combination of characters of $L(n)$