G = compact Lie group

V = f.d. complex representation of G

Lemma There exists a G-invariant positive definite Hermitian form on V.

Recall. $\langle x, y\rangle = \langle gx, gy\rangle$ for all $g \in G$.

Proof: Start with some Hermitian form $\langle x,y\rangle$ on V (positive definite)

$$\frac{1}{\mathrm{Vol}(G)}\int_G \langle gx, gy\rangle \, dg =: \langle x, y\rangle_{new}$$

$G = SU(2) = S^3$

$\int_G \cdots = \int_{S^3} \cdots$

Note: There is a unique (up to scalar) G-invariant volume form on G.

Note: This is very similar to $\frac{1}{|G|}\sum_{g\in G}$ for finite groups.

Note: It is important that G is compact so that integral converges!

Now we claim $\langle\, ,\, \rangle_{new}$ is G-invariant & positive definite.

$$\langle x, x\rangle_{new} = \frac{1}{\mathrm{Vol}(G)}\int_G \langle gx, gx\rangle \, dg$$

$\langle gx, gx\rangle > 0$ for all g

Integral of a positive function is positive.

$$\langle hx, hy\rangle_{new} = \frac{1}{\mathrm{Vol}(G)}\int_G \langle ghx, ghy\rangle \, dg = \text{change variable } g' = gh$$

$$= \frac{1}{\mathrm{Vol}(G)}\int_G \langle g'x, g'y\rangle \, dg' = \langle x, y\rangle_{new}.$$

$$= \frac{1}{\mathrm{Vol}(G)} \int_G \langle g'x, g'y \rangle \, dg' = \langle x, y \rangle_{new}.$$

$\blacksquare$

Thm G = compact Lie group, then any finite-dim representation of G is a direct sum of irreducibles.

Proof: V = some rep of G. If V irreducible, we are done. Otherwise $U \subset V$ is a G-invariant subspace.

By Lemma, choose a G-invt. positive definite Hermitian form in V $\langle , \rangle$

Then $U^\perp$ is G-invariant.

$$U^\perp = \{ y : \langle x, y \rangle = 0 \text{ for } \underline{\text{all}} \ x \in U \}$$

$$\langle x, gy \rangle \underset{\text{form is } G\text{-invt.}}{=} \langle g^{-1}x, y \rangle = 0 \quad \text{since } g^{-1}x \in U$$

$$\Rightarrow gy \in U^\perp.$$

$\langle , \rangle$ is non-degenerate $\Rightarrow U \oplus U^\perp \simeq V$ and continue. $\blacksquare$

Thm Any complex f.d. representation of Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is a direct sum of irreducibles.

Weyl's "unitary trick"

Proof V = rep. of $\mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{su}_2 \otimes \mathbb{C}$

so V is a rep. of $\mathfrak{su}_2$

$SU(2)$ is simply connected ($\pi_1 = 0$) $\Rightarrow$ any fd im. representation of $\mathfrak{su}_2$ lifts to a representation of $SU(2)$

$SU(2)$ is compact ($= S^3$) $\Rightarrow$ by Thm above any representation is a direct sum of irreducibles. $\blacksquare$

representation is a direct sum of irreducibles.

Q: How to lift the representation $L(n)$ to $SU(2)$ or $SL(2,\mathbb{C})$?

Idea $SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$

$\Rightarrow SL(2,\mathbb{C})$ acts on the space of functions on $\mathbb{C}^2$

$$A.\ f(x,y) \longrightarrow f\left(A^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right) \qquad A(B(f)) = A\left(f\left(B^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)\right)$$

$$A \in SL(2,\mathbb{C}) \qquad = f\left(B^{-1}A^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

V_n = space of homogeneous polynomials of degree n in x and y =

$= \mathrm{Span}(x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n) \qquad \dim V_n = n+1.$

Clearly, this is a representation of $SL(2,\mathbb{C})$.

Claim The representation of $\mathfrak{sl}_2(\mathbb{C})$ corresponding to V_n is isomorphic to $L(n)$.

Cor V_n is irreducible, any representation of $SL(2,\mathbb{C})$ is a direct sum of V_{n_i}.

Proof $f \longrightarrow f\left(A^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$

$$H:\ A = \exp(tH) \Rightarrow f(e^{-t}x, e^{+t}y) \xrightarrow[\frac{d}{dt}]{} -x\cdot e^{-t}\cdot\frac{\partial f}{\partial x}(e^{-t}x, e^{+t}y)$$

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad A^{-1} = \begin{pmatrix}e^{-t} & 0\\ 0 & e^{t}\end{pmatrix} \qquad + y e^{+t}\cdot\frac{\partial f}{\partial y}(e^{-t}x, e^{+t}y)$$

$$\xrightarrow{t=0} -x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

$E:\ E = \begin{pmatrix}0 & 1\\ \cdot & \cdot\end{pmatrix} \quad e^{tE} \begin{pmatrix}1 & t\\ \cdot & \cdot\end{pmatrix} = A$

E: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $e^{tE} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = A$

$$A^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$

$$f \longrightarrow f(x-ty, y) \xrightarrow{\frac{d}{dt}} -y\frac{\partial f}{\partial x}(x-ty, y) \xrightarrow{t=0} -y\frac{\partial}{\partial x}$$

F: $e^{tF} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$

$$f \longrightarrow f(x, -tx+y) \xrightarrow{\frac{d}{dt}} -x\frac{\partial f}{\partial y}(x, y-tx) \xrightarrow{t=0} -x\frac{\partial}{\partial y}.$$

<u>Thus</u> The Lie algebra sl_2 acts on the space of functions on $\mathbb{C}^2$ by $H \longrightarrow -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

$$E \longrightarrow -y\frac{\partial}{\partial x} \qquad F \longrightarrow -x\frac{\partial}{\partial y}.$$

On V_n:

$$0 \xleftarrow{F=0} x^n \xleftarrow{F} x^{n-1}y \longleftarrow x^{n-2}y^2 \longleftarrow \cdots \longleftarrow \cdots \xleftarrow{F} y^n \xrightarrow{E} 0$$

$$E(y^n) = 0 \qquad H(x^a y^b) = -ax \cdot x^{a-1}y^b + by \cdot x^a y^{b-1} =$$
$$= (b-a)x^a y^b.$$

So this is an eigenbasis for H with eigenvalues $b-a = n, n-2, \ldots\ldots -n$.

<u>Rmk</u> $\mathrm{Span}(F^k y^\lambda) = \Delta(\lambda)$